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Lecture 21 (10/13/06)

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Graph Sketching

We now put together all the information we can get for the graph of a function and thus have a clearer idea of how the graph should look.

Example. Let $f(x) = \frac{x^2+1}{x^2-1}$, find all (if they exist) horizontal and vertical asymptotes, intervals where the function is increasing, where it is decreasing, where the graph of f is concave up or concave down and points of inflection.

$$\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$$

$$= 1$$

$$\lim_{x\to-\infty} \frac{x^2+1}{x^2-1} = \lim_{x\to-\infty} \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$$

$$= 1.$$

The line $y = 1$ is a horizontal asymptote to the graph of f at both ends of it.
Also $f(x) = \frac{x^2+1}{(x-1)(x+1)}$, so we look at:

$$\lim_{x\to-1-} \frac{x^2+1}{x^2-1} = \lim_{x\to-1-} \frac{x^2+1}{(x-1)(x+1)}$$

$$= \infty$$

$$\lim_{x\to-1+} \frac{x^2+1}{(x-1)(x+1)} = -\infty$$

$$\lim_{x\to1-} \frac{x^2+1}{(x-1)(x+1)} = -\infty$$

$$\lim_{x\to1+} \frac{x^2+1}{(x-1)(x+1)} = \infty.$$

Note that just one "sided infinite limit " is enough to have a vertical asymptote. However, what we did provides more information. We have that $x = -1, x = 1$ are vertical asymptotes.

Now for the intervals where the function is increasing or decreasing. Remember that $-1, 1$ are NOT in the domain of f.

$$f'(x) = -\frac{4x}{(x^2-1)^2}$$

The conclusion from this is that: f is increasing on $(-\infty, -1)$ and on $(-1, 0]$. f is decreasing on $[0, 1)$ and on $(1, \infty)$.

Zero is a point of local maximum.

Concavity:

$$\begin{aligned} f''(x) &= -\frac{4(x^2-1)^2 - 16x^2(x^2-1)}{(x^2-1)^4} \\ &= \frac{12x^4 - 8x^2 - 4}{(x^2-1)^4}. \end{aligned}$$

We now must factor $12x^4 - 8x^2 - 4.$ if we let $u = x^2$ then $u \geq 0$ and we must factor:

$$12u^2 - 8u - 4 = 4(3u+1)(u-1).$$

The roots of this quadratic polynomial in u are $u = 1, u = -\frac{1}{3}$. The last root, namely $u = -\frac{1}{3}$ is not relevant to our problem since $u \geq 0$. The roots of $12x^4 - 8x^2 - 4$ are $-1, 1$.

We have:

$$f''(x) = \frac{4(x-1)(x+1)(3x^2+1)}{(x^2-1)^4}.$$

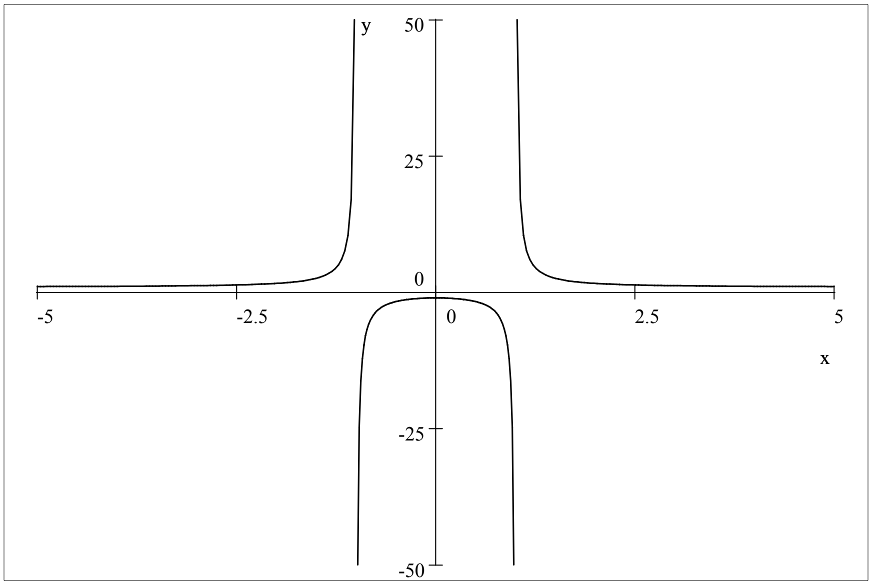
Thus:

$f''(x) > 0$ on $(-\infty, -1)$ and on $(1, \infty)$, graph is concave up.

$f''(x) < 0$ on $(-1, 1)$, graph is concave down.

The points $-1, 1$ are NOT points of inflection since they are NOT in the domain of f (even though the concavity changes from one side to the other.

The graph of f is:



Now we explore the following question: Given an interval I and a function $f : I \to \boldsymbol{R}$, are there points in I where the function attains its highest value or its smallest value?

Of course in this generality we should expect that there is no guarantee of the existence of these points.

Definition. Let I be an interval, $f : I \to \boldsymbol{R}$ a function and $x_0 \in I.$ we say that:

a) x_0 is a point of absolute maximum for f (on I) if, for all $x \in I$, we have:

$$f(x) \leq f(x_0).$$

$f(x_0)$ is the maximum value of f on I.

b) x_0 is a point of absolute minimum for f (on I) if, for all $x \in I$, we have:

$$f(x) \geq f(x_0).$$

$f(x_0)$ is the minimum value of f on I.

c) x_0 is a point of absolute, or global, extremum if it is either a point of absolute maximum or of absolute minimum.

For example, if we consider the function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = x^2,$ then we know that (in this domain) f has no point of absolute maximum and that 0 is a point of absolute minimum. On the other hand, if $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = e^x,$ then there are no points of absolute extremum.

If $I = [a, b]$ and $f : [a, b] \to \boldsymbol{R}$ is continuous then we have that these points are

GUARANTEED to exist.

Theorem. Let $a, b \in \boldsymbol{R}, a < b, f : [a, b] \to \boldsymbol{R}$ is **continuous** then there exist at least one point of absolute maximum and at least one point of absolute minimum for f on $[a, b]$.

Notice that the function in the above Theorem is assumed to be continuous. If f is NOT continuous, then the result is NOT TRUE.

Example. Consider the function $f : [-2, 3] \to \boldsymbol{R}, f(x) = e^x$. Then, since we know that f is continuous, it is guaranteed that it has a point of absolute maximum and a point of absolute minimum. Further more we know that the function is increasing, so:

$$f(-2) = e^{-2} \leq f(x) = e^x \leq f(3) = e^3.$$

Thus -2 is a point of absolute minimum on $[-2, 3]$ and 3 is a point of absolute maximum on $[-2, 3]$.

What can we do to find these points that we know exist (under the conditions of the Theorem)?

If $a, b \in \boldsymbol{R}, a < b, f : [a, b] \to \boldsymbol{R}$ is continuous, what can we do to locate the points of absolute extremum?

The following procedure will give us the answer (in the settings of this course): we always keep in mind that these points may be the endpoints of the interval. If they are not among the endpoints then they must be in the open interval (a, b) and, in this case, they are points of relative extremum as well. Thus the procedure is a follows:

1) Find all the critical points in (a, b).

2) Evaluate $f(a), f(b)$ and also evaluate $f(x_0)$ for ALL critical points of f on (a, b).

3) The points where the value of f is the largest (and there might be more than one) are points of absolute maximum. The points where the value of f is the smallest (and there might be more than one) are points of absolute minimum.

Example. Find all points of absolute extremum for $f : [-2, 3], f(x) = xe^{-x^2}$ and determine their nature.

In order not to forget the end points we evaluate f at them:

$$f(-2) = -2e^{-4} < 0$$
$$f(3) = 3e^{-9} > 0.$$

Now we find the critical points of f in $(-2,3)$.

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2}$$
$$= (1-2x^2)e^{-x^2}$$

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The critical points are:

$$-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$$

Also notice that f is decreasing on $[-2, -\frac{1}{\sqrt{2}}]$, increasing on $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and decreasing on $[\frac{1}{\sqrt{2}}, 3]$.

We evaluate f at these points:

$$f(-\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} < f(-2)$$
$$f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}} > f(3).$$

Thus:

$$f(-\frac{1}{\sqrt{2}}) < f(-2) < 0 < f(3) < f(\frac{1}{\sqrt{2}}).$$

The conclusion is that $-\frac{1}{\sqrt{2}}$ is a point of absolute minimum, $\frac{1}{\sqrt{2}}$ is a point of absolute maximum. $f(-\frac{1}{\sqrt{2}})$ is the minimum value of f and $f(\frac{1}{\sqrt{2}})$ is the maximum value of f on $[-2,3]$.