22M:017, Fall 2006
Lecture 3 (8/25/06)
Juan A. Gatica

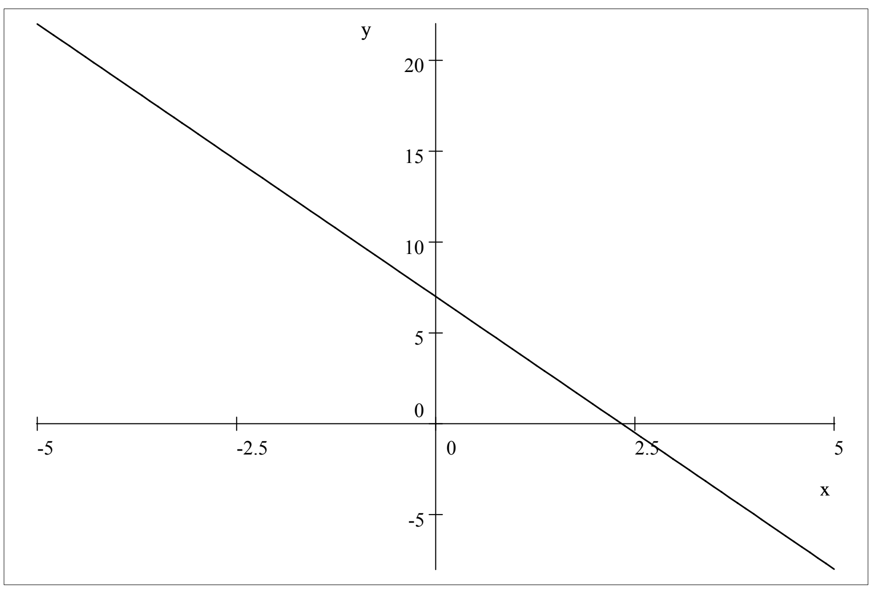
You may want to remember the following formulas:

Volume of a sphere of radius r, $\frac{4}{3}\pi r^3$, surface area of the sphere is $4\pi r^2$, the lateral area of a right circular cylinder of radius r and height $h, 2\pi r^2 h$.

These formulas will come in handy when dealing with verbally stated problems.

Linear Functions

If we look at straight lines in the Cartesian plane, we realize that, unless they are a vertical line, they meet the vertical line test and so all non-vertical lines are the graph of a function.



A non-vertical line in the Cartesian plane

Thus it makes sense to try to find the functions that have graphs that are straight lines.

We first will describe functions that have graphs that are straight lines.

Linear Functions

These are the functions $f : \boldsymbol{R} \to \boldsymbol{R}$ of the form:

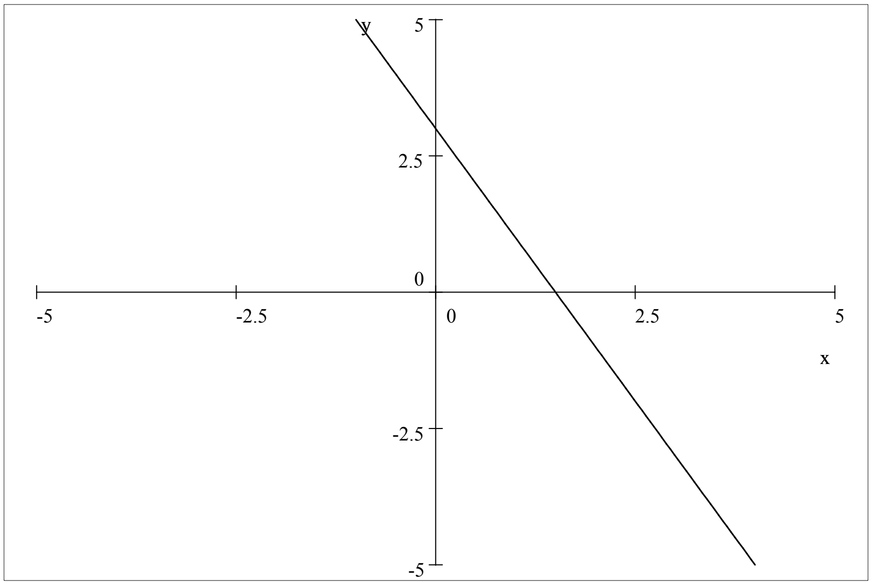
$$f(x) = kx + b$$

where $k, b \in \boldsymbol{R}$ are given real numbers. The graph of a linear function is a straight line, so they are extremely easy to graph.

Example 1. Plot the graph of $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = -2x + 3$ (here $k = -2, b = 3$).

To do this we only need two points in the straight line and we observe that $f(0) = 3, f(1) = 1$, obtaining the points $(0,3),(1,1)$ in the line.

Then the graph is:



If we have a non vertical straight line in the Cartesian plane and we pick two points in it, $(x_1,y_1),(x_2,y_2),x_1 \neq x_2$, then the slope m of the line is given by the quotient of "rise over run ":

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

It follows from elementary analytic geometry that this quotient has the same value for any two different points chosen on the line.

If $m > 0$, the line "ascends " (the values are increasing), if $m = 0, y = b$ for all x and the linear function is constant (the graph is a horizontal straight line), if $m < 0$ the line "descends " (the values are decreasing.

For a linear function $f(x) = kx + b, f(0) = b$ and the slope m of the straight line that is the graph is obtained by choosing $x_1 < x_2$ (because we want it this way) so we have the points $(x_1, kx_1 + b),(x_2, kx_2 + b)$.

$$\begin{aligned} m &= \frac{kx_2 + b - (kx_1 + b)}{x_2 - x_1} \\ &= \frac{kx_2 - kx_1}{x_2 - x_1} \\ &= \frac{k(x_2 - x_1)}{x_2 - x_1} \\ &= k. \end{aligned}$$

Thus the slope is m, the y - intercept (the value of the second coordinate of the point where he graph crosses the $y-$ axis) is $b.$ This is the reason for calling an equation of the form $y = mx + b$ the slope-intercept form of an equation for a straight line.

Facts: two non-vertical straight lines of equation $y = m_1x + b_1, y = m_2x + b_2$ are parallel if and only if $m_1 = m_2$.
mutually perpendicular if and only if $m_1m_2 = -1$.

Example 2. Find an equation for the straight line parallel to the line with equation $y = \frac{2}{3}x + \sqrt{2}$ passing through the point $(3,-1)$.

Here we have information that yields the slope of the line we are searching for as well as a specific point in the line. Thus $m = \frac{2}{3}$, and the equation must be of the form

$$y = \frac{2}{3}x + b.$$

We have to determine the value of $b.$ to do this we use the information that the point $(3,-1)$ is on the line so the coordinates of the point must satisfy its equation:

$$-1 = \frac{2}{3} \times 3 + b$$
$$-1 = 2 + b$$
$$b = -3.$$

The equation we are searching for is, therefore:

$$y = \frac{2}{3}x - 3.$$

Example 3. Find an equation for the line passing through the point $(2,3)$ and perpendicular to the line with equation $y = -\frac{4}{5}x + 2$.

We proceed, as before, to find the slope of the line and hen use the coordinates of the point to find the y-intercept.

The slope m must satisfy:

$$m(-\frac{4}{5}) = -1$$
$$m = \frac{5}{4}.$$

The equation has the form:

$$y = \frac{5}{4}x + b.$$

The point $(2,3)$ is on the line, obtaining:

$$
\begin{aligned}
3 &= \frac{5}{4} \times 2 + b \\
3 &= \frac{5}{2} + b \\
b &= 3 - \frac{5}{2} \\
b &= \frac{1}{2}.
\end{aligned}
$$

The equation we are searching for is:

$$y = \frac{5}{4}x + \frac{1}{2}.$$

We seek an equation for a line when we know two points in it, say $(x_1,y_1),(x_2,y_2)$. If the points have different first coordinates (that is, if $x_1 \neq x_2$), then we get the slope m from the two points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Now we have two choices: we can use one of the points that are in the line to get the intercept or we can remember that the slope does no depend on the particular choice of the two (different) points used to compute it; in this case we see that if (x,y) is any point in the line different from (x_1,x_2) then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}.$$

Then we get:

$$
\begin{aligned}
\frac{y - y_1}{x - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\
y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).
\end{aligned}
$$

The last equation is called the two-point equation for the line.

Example 4. Find an equation for the line that passes through the points $(1,3),(3,-1)$.

Here we label $(x_1,y_1) = (1,3),(x_2,y_2) = (3,-1)$. The slope is:

$$
\begin{aligned}
m &= \frac{-1-3}{3-1} \\
&= \frac{-4}{2} \\
&= -2.
\end{aligned}
$$

The two point equation is:

$$y - 3 = -2(x - 1).$$

Example 5. Consider the equation

$$y - 4 = \pi(x + 1).$$

The slope of this line is $\pi,$ and it passes through the point $(-1, 4)$.

What happens if the two points are different but have the same first coordinate? Then we are in the situation that the two points given are $(x_1, y_1), (x_1, y_2), y_1 \neq y_2$.

Then all points in the line have the same first coordinate, the line is vertical, so it is NOT the graph of a function, the vertical line does NOT have slope and an equation for it is:

$$x = x_1.$$

This simply says that the points in the line are of the form:

$$(x_1, y), y \in \boldsymbol{R}.$$

$$x = x_1$$

is the equation for this vertical line.

Example 6. Find an equation for the vertical line through the point $(2, 5)$.

The value of the first coordinate ($x_1 = 2$) is 2 so an equation for it is:

$$x = 2.$$