22M:017, Fall 2006
Lecture 4 (8/28/06)

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Important Notice: You must remember your discussion section number, since you will have to enter it whenever there is a pop quiz. Without it, your attendance and score will not be entered in our records. I do not know your section number.

We start today with the concept of composition of functions (or function of a function). Let X, Y, Z be sets and $f : X \to Y, g : Y \to Z$ be functions. We define the composition of f and $g, g \circ f : X \to Z$ as the function with domain X (the domain of f) and range contained in the range of $g, R(g),$ given by the rule:

$$g \circ f(x) = g(f(x)), x \in X.$$

Example 1. Let $f : \boldsymbol{R} \to \boldsymbol{R}, g : \boldsymbol{R} \to \boldsymbol{R},$ be the functions:

$$f(x) = x^3$$
$$g(x) = x^2 + x + 1.$$

Then:

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x^3) \\ &= (x^3)^2 + x^3 + 1 \\ &= x^6 + x^3 + 1 \\ f \circ g(x) &= f(g(x)) \\ &= f(x^2 + x + 1) \\ &= (x^2 + x + 1)^3 \\ &= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1. \end{aligned}$$

In particular we see that even if both $g \circ f$ and $f \circ g$ are defined, they need not be equal.

Example 2. Let $f(x) = (x^2 + 1)^3 + \sqrt{x^2 + 1}$. Find functions g, h such that

$$f(x) = g \circ h(x) = g(h(x)).$$

Here we readily see that if we choose $h(x) = x^2 + 1, g(x) = x^3 + \sqrt{x}$

$$\begin{aligned} g(h(x)) &= g(x^2+1) \\ &= (x^2+1)^3 + \sqrt{x^2+1}\,. \end{aligned}$$

then we have achieved our objective.

Note that also if we choose $g_1(x) = (x+\frac{1}{2})^3 + \sqrt{x+\frac{1}{2}}\,, h_1(x) = x^2 + \frac{1}{2}$ then also $f(x) = g_1(h_1(x)) \;= g_1(x^2+\frac{1}{2}) = (x^2+\frac{1}{2}+\frac{1}{2})^3 + \sqrt{x^2+\frac{1}{2}+\frac{1}{2}} \;=$

$g_1 \circ h_1(x) = f(x)$.

Thus the solution to the problem is not unique.

Example 3. Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be the linear function given by

$$f(x) = 2x.$$

Then:

$$\begin{aligned} f\circ f(x) &= f(f(x)) \\ &= 2f(x) \\ &= 2^2x. \\ f\circ(f\circ f)(x) &= f(f\circ f(x)) \\ &= 2(f\circ f(x)) \\ &= 2^3x \\ f\circ(f\circ(f\circ f))(x) &= f((f\circ(f\circ f))(x)) \\ &= 2f\circ(f\circ f)(x) \\ &= 2^4x. \end{aligned}$$

In this case we can see a pattern of the values when we continue to use he composition of f with previous compositions of f.

The following example shows one possible use of this type of reasoning.

Example 4. A certain bank offers a savings account with a 4% yearly interest rate, compounded yearly (that is, exactly one year after a deposit into the account the interest is added to the principal). If a customer makes a deposit of x_0 dollars and keeps it in the account for a number of years without any further deposits, how much is in the account a year deposit into the account? Two years? Three years? Four years?

Since the initial deposit is in the amount of x_0 dollars, immediately after a year, the interest is added to the principal, and he amount in the account is

$$x_0 + 0.04x_0 = 1.04x_0.$$

For the end of second year, the principal over which the interest is accrued is $1.04x$ dollars, so the amount immediately after the second year is:

$$\begin{aligned} 1.04x_0 + 0.04(1.04x_0) &= (1+0.04)1.04x_0 \\ &= (1.04)^2 x_0. \end{aligned}$$

For the end of the third year, the interest is accrued by the principal at the beginning of the year, so we have:

$$\begin{aligned} (1.04)^2 x_0 + 0.04(1.04)^2 x_0 &= (1.04)(1.04)^2 x_0 \\ &= (1.04)^3 x_0. \end{aligned}$$

For the end of the fourth year (now it is no surprise):

$$(1.04)^4 x_0.$$

Define he function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = 1.04x.$ Then we have

$$\begin{aligned} f(x) &= 1.04x \\ f \circ f(x) = f(f(x)) &= 1.04 f(x) = 1.04 \times 1.04x = (1.04)^2 x \\ f \circ (f \circ f)(x) = f(f \circ f(x)) &= 1.04 \times (1.04)^2 x = (1.04)^3 x \\ f \circ (f \circ (f \circ f))(x) = f(f \circ (f \circ f)(x)) &= 1.04 \times (1.04)^3 x = (1.04)^4 x. \end{aligned}$$

In general, if we define the identity function $I : \boldsymbol{R} \to \boldsymbol{R}, I(x) = x,$ then we set:

$$\begin{aligned} f^{*0} &= I \\ f^{*1} &= f \circ f^{*0} = f \\ f^{*2} &= f \circ f^{*1} = f \circ f \\ f^{*3} &= f \circ f^{*2} = f \circ (f \circ f) \end{aligned}$$

$$f^{*(n+1)} = f \circ f^{*n}$$

we see that, by setting $x_n = f^{*n}(x_0)$, then we have a sequence such that each term is the amount in the account after a certain number of years:

$x_0 =$ initial amount

$x_1 =$ amount after one year

$x_2 =$ amount after two years

$x_n =$ amount after n years.

This is an example of an iterative system (also of a discrete dynamical system).

Example 5. A certain medicine has the property that if an amount is given to a certain patient, 24 hours later 70% of the amount remains in the body of the patient. A patient is prescribed a dosage of x_0 milligrams of the medicine to be taken every 24 hours. We would like to know how much medicine is in the patient after any number of days of treatment.

To do this let us first observe that by letting x_n be the amount of medicine in the patient after n days of treatment, we have:

$$\begin{aligned} x_1 &= x_0 + 0.7x_0 \\ x_2 &= x_0 + 0.7x_1 = x_0 + 0.7x_0 + (0.7)^2 x_0 \\ &= (1 + 0.7 + (0.7)^2)x_0 \\ x_3 &= x_0 + 0.7x_2 = (1 + 0.7 + (0.7)^2 + (0.7)^3)x_0 \\ x_4 &= x_0 + 0.7x_3 = (1 + 0.7 + (0.7)^2 + (0.7)^3 + (0.7)^4)x_0. \end{aligned}$$

If we define $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = x_0 + 0.7x$, then:

$$\begin{aligned} f(x_0) &= x_0 + 0.7x_0 = x_1 \\ f^{*2}(x_0) &= f(f(x_0)) = x_0 + 0.7f(x_0) = x_0 + 0.7(x_0 + 0.7x_0) = x_2 \\ f^{*3}(x_0) &= f(f^{*2}(x_0)) = x_0 + 0.7x_2 = x_3 \\ &. \\ &. \\ f^{*n}(x_0) &= (1 + 0.7 + (0.7)^2 + \ldots + (0.7)^n)x_0 \\ f^{*n+1}(x_0) &= f(f^{*n}(x_0)) = (1 + 0.7 + (0.7)^2 + \ldots + (0.7)^n + (0.7)^{n+1})x_0. \end{aligned}$$

We have found a "neat " way of expressing the amount of medicine present in the body of he patient after n days of treatment.

To proceed further we will have to be able to sum a geometrical progression. Let $r > 0, r \neq 1$ and n a positive integer. We want to find a way to evaluate:

$$1 + r + r^2 + r^3 + \ldots + r^n.$$

We call this quantity S_n (it is just a real number). Then we have:

$$\begin{aligned} S_n &= 1 + r + r^2 + r^3 + \ldots + r^n \\ rS_n &= r + r^2 + r^3 + \ldots + r^n + r^{n+1} \\ S_n - rS_n &= 1 - r^{n+1} \\ (1-r)S_n &= 1 - r^{n+1} \\ S_n &= \frac{1 - r^{n+1}}{(1-r)}. \end{aligned}$$

In our particular example $r = 0.7$ and we get:

$$
\begin{aligned}
x_1 &= 1.07x_0 \\
x_2 &= \frac{1-(0.7)^3}{0.3}x_0 \\
x_3 &= \frac{1-(0.7)^4}{0.3}x_0 \\
&. \\
&. \\
&. \\
x_n &= \frac{1-(0.7)^{n+1}}{0.3}x_0.
\end{aligned}
$$

We observe that as n gets big, $(0.7)^{n+1}$ gets very close to zero so the number x_n gets very close (in fact, as close as we wish) to $\frac{1}{0.3}x_0 = (3.3333\ldots)x_0$.

As a special case, if $x_0 = 50$,

$$
\begin{aligned}
x_1 &= (1.7)50 = 85.0 \\
x_2 &= \frac{1-(0.7)^3}{0.3}50 = 109.5 \\
x_3 &= \frac{1-(0.7)^4}{0.3}50 = 126.65
\end{aligned}
$$

and the amount $x_n = \frac{1-(0.7)^{n+1}}{0.3}50$ gets as close as we wish to $\frac{50}{0.3} = 166.67$.

Notice: There is a typo in example 1.4.2 in the text. Please disregard the example in the text and consider carefully Example 5 above instead.