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Lecture 7 (9/6/06)

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More on compounded interest:

The interest earned can be added to the principal any number of times a year, say k times. If the yearly interest rate is $100r\%$ and the original deposit (principal) is of P dollars, the after the first period ($\frac{1}{k}$ of a year), then the amount that earns interest for the second period is $P(1+\frac{r}{k})$, and at the end of the year the account has $P(1+\frac{r}{k})^k$ dollars.

At the end of the t^{th} year the account has $P(1+\frac{r}{k})^{kt}$ dollars.

For example, if a savings account offers an interest rate of 5% (so $100r = 5$, or $r = 0.05$) and compounds it monthly, so $k = 12$, and an account is established with a deposit of $1000, then after one year the account has

$$1000(1+\frac{0.05}{12})^{12} \cong 1051.2$$

dollars and after 3 years it has:

$$1000(1+\frac{0.05}{12})^{36} \cong 1161.5$$

dollars.

What if the interest is added to the principal the moment it is earned? We think of this situation in the following way: we compound the interest k times a year and let k grow without bound, and if this sequence of compounded amounts has a limit, then that would be the equivalent of adding the interest to the principal the moment it is earned, for short it is called continuously compounded interest.

The following is the result of the process:

$$\lim_{k\to\infty} P(1+\frac{r}{k})^{kt} = Pe^{rt}.$$

Example. An account is opened with an initial deposit of P dollars at an annual interest rate of 5% compounded continuously. Find when the amount in the account will double and find out the actual percentage of growth of the account for the first year (effective interest rate).

The amount in the account after t years is

$$Pe^{0.05t}.$$

For the first question we want to find t_0 such that

$$\begin{aligned}
Pe^{0.05t_0} &= 2P \\
e^{0.05t_0} &= 2 \\
0.05t_0 &= \ln(2) \\
t_0 &= \frac{\ln(2)}{0.05} \\
t_0 &\cong 13.863 \text{ years.}
\end{aligned}$$

The amount in the account at the end of the first year is :

$$Pe^{0.05} \cong P(1.0513).$$

The account grew 5.13% during the first year.

To finish our pre-calculus review, let $r > 0$,$P > 0$ and consider functions of the type $f(t) = Pe^{rt}$ and of the type $f(t) = Pe^{-rt}$. The first ones are called exponential growth functions (for example bank accounts with continuously compounded interest) while the second type of functions are called exponential decay functions (used, among other things, to study radioactive decay, the amount of a drug present after in a patient after a number of hours (or days) have elapsed.

For functions of exponential growth, one of the first question is : Is there a fixed amount of time for which the value of the function doubles, no matter what P is? The answer is yes and that period of time is called the "doubling time ". To get it, we have to answer the question: is there t_0 such that $f(t_0) = 2P$, and t_0 must not depend on P?

If the function is $f(t) = Pe^{rt}$, we must solve:

$$\begin{aligned}
Pe^{rt_0} &= 2P \\
e^{rt_0} &= 2 \\
rt_0 &= \ln(2) \\
t_0 &= \frac{\ln(2)}{r}.
\end{aligned}$$

Definition. $t_0 = \frac{\ln(2)}{r}$ is called the doubling time for $f(t) = Pe^{rt}$.

We see that t_0 has these properties,so we have answered the question.

Example. A bank account with yearly interest compounded continuously doubles in 10 years. What is the interest rate in percentages?

Here the function is $f(t) = Pe^{rt}$, where $100r$ is the interest rate in percentages and P is the amount deposited. If t_0 is the doubling time, we have:

$$
\begin{aligned}
Pe^{rt_0} &= 2P \\
e^{rt_0} &= 2 \\
rt_0 &= \ln(2) \\
t_0 &= \frac{\ln(2)}{r} \\
10 &= \frac{\ln(2)}{r} \\
r &= \frac{\ln(2)}{10}
\end{aligned}
$$

The interest rate in percentages is $100\frac{\ln(2)}{10} = 10\ln(2)\%$

For functions of exponential decay, the function is decreasing and now the question is about half-life. If the function is $f(t) = Pe^{-rt}(r > 0)$, is there a time t_0, independent of P such that $f(t_0) = \frac{1}{2}P$? This amount of time is called the half-life.

We proceed as before:

$$
\begin{aligned}
Pe^{-rt_0} &= \frac{1}{2}P \\
e^{-rt_0} &= \frac{1}{2} \\
-rt_0 &= \ln(\frac{1}{2}) \\
-rt_0 &= -\ln(2) \\
t_0 &= \frac{\ln(2)}{r}.
\end{aligned}
$$

Example. The amount of drug in a certain patient undergoes exponential decay with a half-life of 2 hours (Ibuprofen has a half life of less than two hours, but "close "to it). How long after administration of the drug has 30% been eliminated from the body?

Here the function is $f(t) = Pe^{-rt}, 2 = \frac{\ln(2)}{r}, r = \frac{\ln(2)}{2}$, so $f(t) = Pe^{-\frac{\ln(2)}{2}t}$. The question now is: When is $f(t) = 0.7P$.

$$
\begin{aligned}
Pe^{-\frac{\ln(2)}{2}t} &= 0.7P \\
e^{-\frac{\ln(2)}{2}t} &= 0.7 \\
-\frac{\ln(2)}{2}t &= \ln(0.7) = \ln(7) - \ln(10) \\
\frac{\ln(2)}{2}t &= \ln(10) - \ln(7) \\
t &= \frac{2(\ln(10) - \ln(7))}{\ln(2)} \quad \text{(this is the answer I want)}
\end{aligned}
$$

'For a decimal approximation we have $t = \frac{2(\ln(10)-\ln(7))}{\ln(2)} = 1.0291$. Try to convert this to hours, minutes and seconds.

Example. A savings account compounds yearly interest continuously in such a way that two years after depositing an amount, the principal has grown by 8%. Find the doubling time for this account.

Since the interest is compounded continuously, the principal follows the exponential growth model, so if $P(t)$ is the principal at t years after deposit, then:

$$P(t) = Pe^{\alpha t}, t \geq 0,\ P \text{ is the original amount of deposit.}$$

In our setting we do not know the value of P but this is not a problem. We need to know the value of α, which was not given directly to us. However, we were given information to find it:

$$\begin{aligned} P(2) &= 1.08P \\ Pe^{2\alpha} &= 1.08P \\ e^{2\alpha} &= 1.08 \\ 2\alpha &= \ln(1.08) \\ \alpha &= \frac{\ln(1.08)}{2}. \end{aligned}$$

Now we can find the doubling time t_0 :

$$\begin{aligned} t_0 &= \frac{\ln(2)}{\left(\frac{\ln(1.08)}{2}\right)} \\ &= \frac{2\ln(2)}{\ln(1.08)} \text{ years, and this is the answer I will be looking for.} \end{aligned}$$

An approximation is : $t_0 \cong 18.013$ years.

Example. A drug in a patient is undergoing exponential decay with a half life of 5 hours. How long will it take for a dose to diminish by 32%?

Here $f(t) = Ke^{-\alpha t}$ and we want to know when there will be $0.68K$ left in the patient. We are not given α directly but we can get it from the half-life information:

$$\begin{aligned} \frac{\ln(2)}{\alpha} &= 5 \\ \alpha &= \frac{\ln(2)}{5}. \end{aligned}$$

Now we know that $f(t) = Ke^{-\frac{\ln(2)}{5}t}$. We need to solve:

$$Ke^{-\frac{\ln(2)}{5}t} = 0.68K$$

$$e^{-\frac{\ln(2)}{5}t} = 0.68$$

$$-\frac{\ln(2)}{5}t = \ln(0.68)$$ (this is a negative number)

$$t = -\frac{5\ln(0.68)}{\ln(2)}$$ hours.

An approximation is : $t = 2.7820$ hours.