22M:017, Fall 2006
Lecture 9 (9/11/06)
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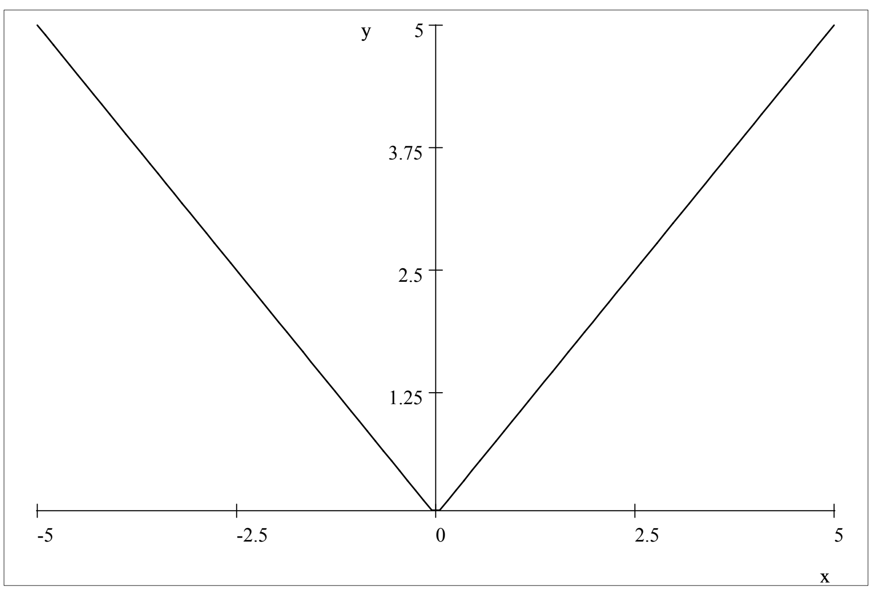
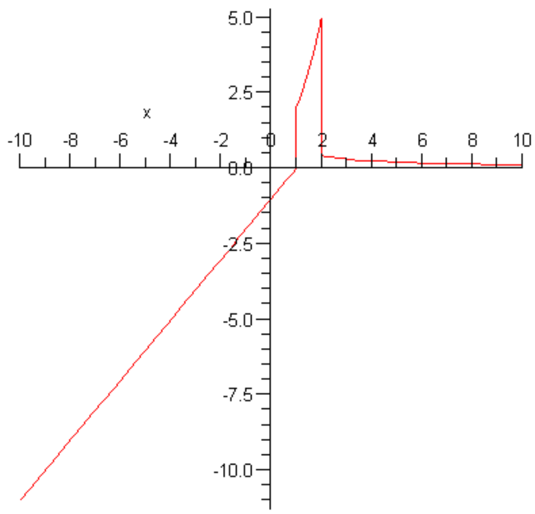
Important Notice: there are quite a few students that have not entered correctly their information in pop quizzes, so we will make adjustments for this fact (no penalties for the first two pop quizzes). From now on it will be the student's responsibility to enter their information correctly: their student ID number and their section number, using the jkl columns in the answer form. The forms should be filled using a #2 lead pencil, so please make sure to bring one with you.

Please visit the website where the syllabus is. There was an addition to problems from Section 2.3.

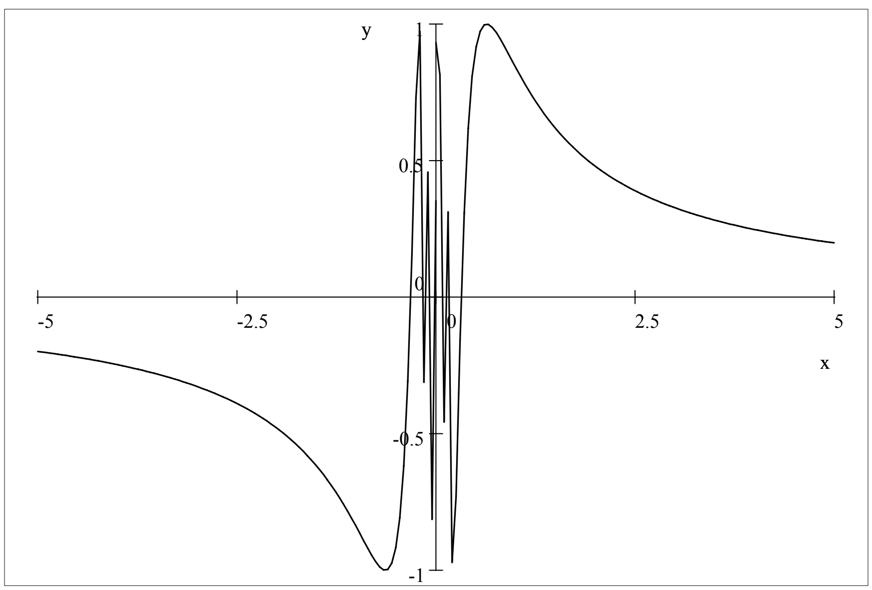
It is important to know that when dealing with limits, the functions may be defined as piecewise, and that looking at both sides of the real number in the domain is important. For example, we may have graphs such as:



or:



Vertical lines are NOT part of graph



In both cases we see that if we approach the point of interest from one side we approach a limit, and if we approach the point from the other side, we also have a limit. In the first picture, both of these limits exist and are equal, so we can conclude that the FULL limit exists and is equal to the "sided "limits, while in the second picture the sided limits are different and it follows that the full limit does not exist.

Definition. Suppose that a function f is defined in an interval of he form (a, x_0). Then we are interested in what happens when we approach x_0 from the left, and we say that L is the limit of f as x approaches x_0 from the left, written as $\lim_{x \to x_0 -}$, if the values of $f(x)$ get as close as we wish to L by taking x close to x_0 but smaller than x_0.

$$L = \lim_{x \to x_0-} f(x).$$

Example. Find $\lim_{x \to -1-} \sqrt{x^2 - 1}$.

Notice that $f(x) = \sqrt{x^2 - 1}$, so the "natural domain "of f is $(-\infty, -1] \cup [1, \infty)$. f cannot be evaluated for $x > -1$, near -1. It is "clear " that $\lim_{x \to -1-} \sqrt{x^2 - 1} = 0$.

Example. Find $\lim_{x \to 1-} \frac{e^{\sqrt{1-x^2}}}{x^2+x+1}$.

In this case the function

$$f(x) = \frac{e^{\sqrt{1-x^2}}}{x^2 + x + 1}$$

and its natural domain is $[-1, 1]$. The denominator has limit 3 when x approaches 1, and $\lim_{x \to 1-} \sqrt{1 - x^2} = 0$. Thus:

$$\lim_{x \to 1-} \frac{e^{\sqrt{1-x^2}}}{x^2 + x + 1} = \frac{e^0}{3} = \frac{1}{3}.$$

As we did with limits from the left we can also deal with limits from the right.

Definition. Let x_0 be a real number and f a function defined in an interval of the form (x_0, a). We say that a real number L is the limit of f as x approaches x_0 from the right if the values of $f(x)$ get as close to L as we wish by taking x close to x_0 but greater that x_0.

$$L = \lim_{x \to x_0+} f(x).$$

Example. Find $\lim_{x \to 1+} \sqrt{x^2 - 1}$. Notice that $f(x) = \sqrt{x^2 - 1}$, so the "natural domain "of f is $(-\infty, -1] \cup [1, \infty)$. f cannot be evaluated for $x < 1$, but near 1. It is "clear " that $\lim_{x \to 1+} \sqrt{x^2 - 1} = 0$.

Example.

$$\lim_{x \to 0+} \frac{|x|}{x} = 1$$

since

$$\frac{|x|}{x} = 1, x > 0$$

$$\frac{|x|}{x} = -1, x < 0.$$

Observation: Let $a < b$ be real numbers, $x_0 \in (a, b)$ and f a function defined for all $x \neq x_0, x \in (a, b)$. Then: $\lim_{x \to x_0} f(x)$ exists if and only if:

$$\lim_{x \to x_0-} f(x), \lim_{x \to x_0+} f(x) \text{ both exist and } \lim_{x \to x_0-} f(x) = \lim_{x \to x_0+} f(x).$$

If either of $\lim_{x\to x_0-} f(x), \lim_{x\to x_0+} f(x)$ does not exist, the "full " limit does not exist, and if both sided limits exist but are different, the full limit does NOT exist.

Example. Let f be the function given by the rule

$$f(x) = \frac{|x|}{x}, x \neq 0$$
$$f(0) = 1.$$

Then, by the argument given above,

$$\lim_{x\to 0-} f(x) = -1$$
$$\lim_{x\to 0+} f(x) = 1.$$

Thus $\lim_{x\to 0} f(x)$ does not exist.

Example. Let f be the function given by the rule:

$$f(x) = e^{2x}, x > 0$$
$$f(x) = 2x + 1, x < 0.$$

Does $\lim_{x\to 0} f(x)$ exist and if it does, what is its value?

Note that 0 is not in the domain of f.

We have:

$$\begin{aligned}\lim_{x\to 0-} f(x) &= \lim_{x\to 0-}(2x+1)\\ &= 1.\\ \lim_{x\to 0+} f(x) &= \lim_{x\to 0+} e^{2x}\\ &= e^0 = 1.\end{aligned}$$

Thus both sided limits exist and are equal. We have proved that

$$\lim_{x\to 0} f(x) = 1.$$

Example. Let $f(x)$ be given by the rule:

$$f(x) = 2x - 1, x < 1$$
$$f(x) = 3x + 1, x \geq 1.$$

Does $\lim_{x\to 1} f(x)$ exist and if it does, what is its value?

In this case we have that 1 is in the domain of f and that $f(1) = 4$:

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(2x-1) = 1$$
$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+}(3x+1) = 4.$$

Since $1 \neq 4$, $\lim_{x\to 1} f(x)$ does NOT exist.

Definition. Let I be an open interval, $x_0 \in I, f : I \to \boldsymbol{R}$ a function (x_0 is in the domain of f). We say that f is continuous at x_0 if two conditions hold: 1. $\lim_{x \to x_0} f(x)$ exists, and 2. $\lim_{x \to x_0} f(x) = f(x_0)$.

Example. Let f be the function given by the rule:

$$f(x) = e^{2x}, x \geq 0$$
$$f(x) = 2x + 1, x < 0.$$

Is f continuous at 0? Notice that now $f(0) = 1$.

We have already showed that

$$\lim_{x \to 0} f(x) = 1 = f(0).$$

f is continuous at 0.

Example. Let f be the function given by the rule:

$$f(x) = 2x - 1, x < 1$$
$$f(x) = 3x + 1, x \geq 1.$$

Is the function continuous at 1?.

No, since $\lim_{x \to 1} f(x)$ does not exist.

Definition. Let I be an open interval and $f : I \to \boldsymbol{R}$ be a function. We say that f is continuous (without specifying the point of interest) if f is continuous at every point of I.

Recall the Rules for Limits:

If f, g are defined in an open interval around a point x_0, but not necessarily defined at this particular point (x_0), and $\alpha \in \boldsymbol{R}$, then, if $\lim_{x \to x_0} f(x) = L_1, \lim_{x \to x_0} g(x) = L_2$:

1. $\lim_{x \to x_0} \gamma = \gamma$.
2. $\lim_{x \to x_0} x = x_0$.
3. $\lim_{x \to x_0} (\alpha f(x)) = \alpha \lim_{x \to x_0} f(x) = \alpha L_1$.
4. $\lim_{x \to x_0} (f(x) g(x)) = \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x) = L_1 L_2$.
5. $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = L_1 + L_2$.
6. If $L_2 \neq 0, \lim_{x \to x_0} (\frac{f(x)}{g(x)}) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{L_1}{L_2}$.
7. If $a > 0, a \neq 1, \lim_{x \to x_0} a^{f(x)} = a^{\lim_{x \to x_0} f(x)} = a^{L_1}$.
8. If $\lim_{x \to L_1} g(x) = L_3$ exists, then $\lim_{x \to x_0} g(f(x)) = L_3$
9. If $x_0 > 0, \lim_{x \to x_0} \ln(x) = \ln(x_0)$.
10. If $x_0 > 0, \lim_{x \to x_0} x^x = x_0^{x_0}$.

A consequence of all of these rules is that now they should be considered as giving rules for continuity, as follows:

1. Any constant function is continuous.
2. The identity function $I(x) = x$ is continuous.
3. If $n \in \mathbf{N},$ then $f(x) = x^n$ is continuous. The constant multiple rule and the sum rule yield AS CONSEQUENCE THAT ANY POLYNOMIAL FUNCTION IS CONTINUOUS.
4. Any exponential function is continuous as well as any logarithmic function is continuous on $(0, \infty)$.

Example. $f(x) = 3x^3 + e^x + \log_3(x)$ is continuous on $(0, \infty)$. This function is not defined for any negative real number and is not defined at zero.

5. The composition of two continuous functions is continuous.

Example. $f(x) = \pi \ln(\frac{1+x^2}{1+x^4})$ is continuous.