Homework 1, Summer 2006.

1. Prove that $2^n > n$ for all natural numbers n.

2., How many pairs of positive integers (m,n) are there such that $m < n < 41$? Remember the formulas for adding up natural numbers that we have seen in class.

3. Prove that if n is any natural number then

$$\sum_{1=1}^{n}(2i-1) = 1+3+5+\ldots+(2n-1) = n^2.$$

The problem is about the sum of the first n odd natural numbers.

4. Define the sequence of numbers $\{F_n\}_{n=0}^{\infty}$ (the Fibonacci sequence of numbers) by

$$F_0 = 0, F_1 = 1 \text{ and, for } n \geq 2,$$
$$F_n = F_{n-1} + F_{n-2}.$$

Compute the first 10 terms of the sequence. Also, prove that if $\alpha = \frac{1}{2}(1+\sqrt{5}), \beta = \frac{1}{2}(1-\sqrt{5})$, then, for $n \geq 0, n$ an integer,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n).$$

Note: If x is any real number, we agree that $x^0 = 1$.

5. For n a natural number and $k \leq n$ a natural number, define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (here we define $0! = 1, (n+1)! = (n+1)n!$). Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

6. Find a formula for $1 + \sum_{j=1}^{n} j!j$ and use mathematical induction to prove it.

7. Prove that

$$\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2.$$

Hint: we have a formula for $\sum_{i=1}^{n} i$.

8. Prove that if $a_1, a_2, \ldots a_n$ are positive numbers, then

$$(\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n}(\frac{1}{a_i}) \geq n^2.$$

9. Prove that if x is a real number such that $1 + x > 0$ then for any positive integer n we have:

$$(1 + x)^n \geq 1 + nx.$$

10. Prove that if $r \neq 1$ is a real number and n is a positive integer, then:

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$