

## Lecture 1, Summer 2006

We will concentrate during our meetings on the topic of linear algebra, which will be the common basis for your mathematical research projects. However we will discuss at the very beginning the necessity of proofs in mathematics and we will do some proofs that use mathematical induction.

Let us start by having some fun with some sums of positive integers, also called natural numbers. In order to make good use of space and for really understanding some statements we will use the following notation: if  $a_0, a_1, \dots, a_n$  are real numbers then we denote by  $\sum_{i=0}^n a_i$  the sum  $a_0 + a_1 + a_2 + \dots + a_n$ .

I will begin by computing sums of all the natural numbers (positive integers) up to a given one, so we perform the following sums:

$$\sum_{i=1}^1 i = 1 = 1$$

$$\sum_{i=1}^2 i = 1 + 2 = 3$$

$$\sum_{i=1}^3 i = 1 + 2 + 3 = 6$$

$$\sum_{i=1}^4 i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=1}^5 i = 10 + 5 = 15$$

$$\sum_{i=1}^6 i = 15 + 6 = 21$$

Do we see a pattern? Perhaps not yet, other than:

$$\sum_{i=1}^2 i = 2 + \sum_{i=1}^1 i$$

$$\sum_{i=1}^3 i = 3 + \sum_{i=1}^2 i$$

$$\sum_{i=1}^4 i = 4 + \sum_{i=1}^3 i$$

$$\sum_{i=1}^5 i = 5 + \sum_{i=1}^4 i$$

$$\sum_{i=1}^6 i = 6 + \sum_{i=1}^5 i.$$

Let us denote

$$S_n = \sum_{i=1}^n i.$$

From the above we see that we proceeded in the following way: we defined  $S_1$  to be one and we then said that if we have a sum defined for some natural number  $n$  the the sum corresponding to the next natural number is the next natural number added to the sum that we started out with. So we have, since  $2 = 1 + 1, S_2 = 2 + S_1 = 2 + 1 = 3$ , and so now that we have  $S_2$  and

$3 = 2 + 1, S_3 = 3 + S_2 = 3 + 3 = 6, S_4 = S_{3+1} = 4 + S_3 = 4 + 6 = 10$  and so on. Hence we can define these numbers in two different ways.

Back to our sums. Would you be surprised if I told you that

$$S_{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55?$$

Well, since you can easily check it, you can confirm that what I told you is true. How about  $S_{100}$ ?

Actually this was a problem that was posed to his class when he was in grade school. The teacher wanted to keep hem occupied for a time, but Gauss responded immediately that the answer was 5050, which is correct. The teacher reportedly asked him how he came up with the answer to which he responded that one hundred plus one is 101, 99 plus 2 is 101. 98 plus three is 101 and, since you can do this fifty times, the answer is  $50 \times 101 = 5050$ . How about that?

From this we may infer that

$$S_n = \frac{n(n+1)}{2}.$$

If we do calculations we can verify this for a great number of cases Is this enough?

Of course you know that I will say "No, it does not prove it". Is my statement capricious? Am I pulling your leg? I hope not.

You see that the statement I made must apply to any positive integer, so that, for example

$$S_{1000000} = \frac{1000000 \times 1000001}{2} = 500\,000\,500\,000.$$

It is not impossible to check it out, but it would take quite a time. If we use huge numbers we would be in a situation that the rest of our lifetime would not give us the time to check it out for one of these huge numbers, so a proof would be very nice indeed. Moreover, there are statements that hold true for quite a few of the natural numbers but fail when a number large enough is tried. An example of this is the following rather easy statement:

If  $n$  is a natural number then  $f(n) = n^2 - n + 41$  is a prime number. Let us check it for some of the first natural numbers.

$$f(1) = 1 - 1 + 41 = 41, \text{ a prime number.}$$

$$f(2) = 2^2 - 2 + 41 = 43, \text{ a prime number.}$$

$$f(3) = 3^2 - 3 + 41 = 47, \text{ a prime number.}$$

$$f(4) = 4^2 - 4 + 41 = 57, \text{ a prime number.}$$

$$f(5) = 5^2 - 5 + 41 = 61, \text{ a prime number.}$$

$$f(6) = 6^2 - 6 + 41 = 71, \text{ a prime number.}$$

$$f(7) = 7^2 - 7 + 41 = 83, \text{ a prime number.}$$

These are quite a few tries, all with the desired result.

We could then try

$$f(20) = (20)^2 - 20 + 41 = 421.$$

A bit of work shows that this is indeed a prime number. Actually we could go on checking all the way to  $f(40) = 40^2 - 40 + 41 = 1601$  and some work would show that this is also a prime number. So we have decided that the statement is true for the first forty natural numbers. Is it then true for all of them?

Well, we have

$$f(41) = (41)^2 - 41 + 41 = (41)^2.$$

This is certainly not a prime number.

Can we find a procedure that will enable us to prove our "conjecture" and many other ones relating to the natural numbers? The answer is YES and it is the method of proof by induction, which can be described as follows:

First Principle of Mathematical Induction. Let  $S(n)$  be a family of statements, where  $n$  indicates a natural number (one statement per natural number).

Suppose that:

1.  $S(1)$  is true.
2. The truth of  $S(n)$  implies the truth of  $S(n + 1)$ .

Then  $S(n)$  is true for every natural number  $n$ .

To get an inkling of why this is true, think of an infinite row of dominoes of the same size that are placed on an infinite table, one behind the other and at a distance smaller than the height of the dominoes. If we knock down the first one towards the second one, then the second one will topple, and this will cause the third one to topple, which causes the fourth one to topple and so on, until all of them have fallen.

We could accept this principle as an axiom to build on. However it is better, in my opinion to use another fact, more easily accepted as an axiom and build from it. We will state it as an axiom. For this we will use what we already know about sets, elements, subsets, etc. and we will denote by  $\phi$  the empty set, that is  $\phi$  is the only set without elements. If  $X$  is a set and  $x$  is an element of it, we write  $x \in X$  to indicate this and we also write  $S \subseteq X$  to indicate that  $S$  is a subset of  $X$ .

**Least Integer Axiom.** Let  $N$  be the set of positive integers, also called the set of natural numbers, that is  $N = \{1, 2, 3, 4, \dots, n, \dots\}$ , and let  $S$  be a non empty subset of  $N$ , that is  $S$  is a set of positive integers that has at least one element. Then there is an element  $n_0$  of  $S$  such that  $n_0 \leq k$  for all elements  $k$  of  $S$  (that is,  $S$  has a first element).

Let us see that the First Principle of Mathematical Induction is a consequence of this axiom. [ ]

Assume that we have a sequence of statements  $\{S(n)\}_{n \in N}$  such that  $S(1)$  is true and if  $S(n)$  is true then so is  $S(n + 1)$ . We want to show that all the statements are true.

Let  $X$  be the set of all positive integers  $n$  for which  $S(n)$  is not true. We want to prove that  $X$  is the empty set, that is, that there are no such positive integers.

We proceed by way of contradiction, so we assume that  $X$  is not empty. Then, by the Least Integer Axiom,  $X$  has a first element  $n_0$ .

By the assumptions of the First Principle,  $n_0 \neq 1$ . We conclude that  $n_0$  is bigger than one, and it follows that  $n_0 - 1$  is also a positive integer and it must be the case that  $S(n_0 - 1)$  is true, since if it were not, then  $n_0 - 1 < n_0$  would be an element of  $X$ , contradicting the fact that  $n_0$  is the smallest element of  $X$ . But now, from the hypothesis, it must be the case that  $S_{n_0}$  is true, contradicting the fact that  $n_0 \in X$ .

The conclusion is that  $X$  must be empty, which is what we wanted to prove.

