Lecture 4

WE have seen that triangular systems are "easy" to solve and we have also seen that given a linear system of two equations in two unknowns we can always find an equivalent triangular system, which we can solve with ease.

We will now use the following operations with systems of equations that will lead us to equivalent systems that are easy to solve (that is, to find their solution set):

Operation 1. Multiply an equation by a nonzero real number.
Operation 2. Add a multiple of one equation to another equation.
Operation 3. Interchange the order of two equations.

These operations always lead us to an equivalent system of linear equations (you have done this work in the last homework) and our objective is to transform a given system into one that is easily solved. The main difficulty with the procedure we are about to learn is not a conceptual one but it an arithmetical one: it is very easy to make algebraic errors when performing the operations with a given system.

The main operations will be the first two and we start with an example:

Find the solution set of the system

$$\begin{aligned} 4x_1 + 2x_2 - 8x_3 + 2x_4 &= 12 \\ 3x_1 + \frac{3}{2}x_2 - 2x_3 + 3x_4 &= 2. \end{aligned}$$

We will perform the operations one by one.

The first operation is to multiply he first equation by $\frac{1}{4}$:

$$\begin{aligned} x_1 + \frac{1}{2}x_2 - 2x_3 + \frac{1}{2}x_4 &= 3 \\ 3x_1 + \frac{3}{2}x_2 - 2x_3 + 3x_4 &= 2 \end{aligned}$$

The second operation is to add -3 times the first equation to the second equation:

$$\begin{aligned} x_1 + \frac{1}{2}x_2 - 2x_3 + \frac{1}{2}x_4 &= 3 \\ 4x_3 + \frac{3}{2}x_4 &= -7 \end{aligned}$$

Now we multiply the second equation by $\frac{1}{4}$:

$$\begin{aligned} x_1 + \frac{1}{2}x_2 - 2x_3 + \frac{1}{2}x_4 &= 3 \\ x_3 + \frac{3}{8}x_4 &= -\frac{7}{4} \end{aligned}$$

The fact that x_2 does not appear in the second equation tells us that x_3 can be any real number.

Now we can write x_3 in terms of x_4.

$$x_3 = -\frac{3}{8}x_4 - \frac{7}{4}$$

We use this in the first equation:

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{7}{2} + \frac{3}{4}x_4 + \frac{1}{2}x_4 &= 3 \\ x_1 + \frac{1}{2}x_2 + \frac{5}{4}x_4 &= -\frac{1}{2} \\ x_1 &= -\frac{1}{2}x_2 - \frac{5}{4}x_4 - \frac{1}{2}. \end{aligned}$$

If x_2 and x_4 are any real numbers,

$$(-\frac{1}{2}x_2 - \frac{5}{4}x_4 - \frac{1}{2}, x_2, -\frac{3}{8}x_4 - \frac{7}{4}, x_4)$$

is a solution of our system. The solution set is:

$$S = \{(\frac{1}{2}x_2 - \frac{5}{4}x_4 - \frac{1}{2}, x_2, -\frac{3}{8}x_4 - \frac{7}{4}, x_4) : x_2, x_4 \in \boldsymbol{R}\}.$$

Example. Find the solution set for the system:

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 1 \\ 2x_2 + x_3 &= 0 \\ -3x_1 + 2x_2 &= 5 \end{aligned}$$

We would like to have 1 as a coefficient of x_1 in the first equation, since then it will be easy to cancel $x_1$from any other equation where it appears. To do this, we multiply the first equation by $\frac{1}{2}$:

$$\begin{aligned} x_1 - \frac{3}{2}x_2 + 2x_3 &= \frac{1}{2} \\ 2x_2 + x_3 &= 0 \\ -3x_1 + 2x_2 &= 5 \end{aligned}$$

Now we add 3 times the first equation to the third equation:

$$\begin{aligned} x_1 - \frac{3}{2}x_2 + 2x_3 &= \frac{1}{2} \\ 2x_2 + x_3 &= 0 \\ -\frac{5}{2}x_2 + 6x_3 &= \frac{13}{2} \end{aligned}$$

Now we would like the second equation to start with x_2 and, to get this, we multiply the equation by $\frac{1}{2}$:

$$
\begin{aligned}
x_1 - \frac{3}{2}x_2 + 2x_3 &= \frac{1}{2} \\
x_2 + \frac{1}{2}x_3 &= 0 \\
-\frac{5}{2}x_2 + 6x_3 &= \frac{13}{2}
\end{aligned}
$$

Now we want to "get rid" of the x_2 appearing in the third equation. We achieve this by adding $\frac{5}{2}$ times the second equation to the third equation:

$$
\begin{aligned}
x_1 - \frac{3}{2}x_2 + 2x_3 &= \frac{1}{2} \\
x_2 + \frac{1}{2}x_3 &= 0 \\
\frac{29}{4}x_3 &= \frac{13}{2}
\end{aligned}
$$

Now we multiply the last equation by $\frac{4}{29}$:

$$
\begin{aligned}
x_1 - \frac{3}{2}x_2 + 2x_3 &= \frac{1}{2} \\
x_2 + \frac{1}{2}x_3 &= 0 \\
x_3 &= \frac{26}{29}.
\end{aligned}
$$

We have:

$$
\begin{aligned}
x_3 &= \frac{26}{29} \\
x_2 + \frac{13}{29} &= 0 \\
x_2 &= -\frac{13}{29} \\
x_1 + \frac{39}{58} + \frac{52}{29} &= \frac{1}{2} \\
x_1 &= \frac{1}{2} - \frac{39}{58} - \frac{52}{29} \\
x_1 &= \frac{29 - 39 - 104}{58} = -\frac{114}{58} = -\frac{57}{29}
\end{aligned}
$$

The solution set is:

$$
S = \{(-\frac{57}{29}, -\frac{13}{29}, \frac{26}{29})\}.
$$

The Gaussian Elimination process is the repeated application of the steps illustrated above, done in an **orderly fashion.**

If the system has the variables $x_1, x_2, x_3, \ldots, x_n,$ then the variable x_1 is eliminated from all equation except one, which is sometimes called the pivot equation for x_1, and, by interchanging equations we get the first equation to be the pivot equation, then x_2 is eliminated from all equations but one (which is sometimes called the pivot equation for

x_2) and, by exchanging equations if necessary, we can always get the second equation to be the pivot equation for x_2. We do the same for the rest of the variables, one by one, and arrive at a triangular system which we can easily solve, with the method of **backsolving,** which is simply solving for x_n then for x_{n-1} and so on.

Example. Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 3x_2 + 2x_3 &= 4 \\ 6x_2 + 4x_3 &= 8 \end{aligned}$$

Here we have that x_1 appears only in the first equation, so we only have to multiply the first equation by $\frac{1}{2}$ to have the one we desire multiplying x_1.

$$\begin{aligned} x_1 - x_2 + \frac{1}{2}x_3 &= \frac{3}{2} \\ 3x_2 + 2x_3 &= 4 \\ 6x_2 + 4x_3 &= 8 \end{aligned}$$

We will multiply the second equation by $\frac{1}{3}$ to get x_2 multiplied by 1 and then we multiply the resulting second equation by -6 and add it to he third equation:

$$\begin{aligned} x_1 - x_2 + \frac{1}{2}x_3 &= \frac{3}{2} \\ x_2 + \frac{2}{3}x_3 &= \frac{4}{3} \\ 0 &= 0 \end{aligned}$$

The third equation disappeared and from the second equation we get that

$$\begin{aligned} x_1 - x_2 + \frac{1}{2}x_3 &= \frac{3}{2} \\ x_2 &= -\frac{2}{3}x_3 + \frac{4}{3} \end{aligned}$$

Now we use the first equation

$$\begin{aligned} x_1 + \frac{2}{3}x_3 - \frac{4}{3} + \frac{1}{2}x_3 &= \frac{3}{2} \\ x_1 + \frac{7}{6}x_3 &= \frac{17}{6} \\ x_1 &= -\frac{7}{6}x_3 + \frac{17}{6}. \end{aligned}$$

The solution set is:

$$S = \{(-\frac{7}{6}x_3 + \frac{17}{6}, -\frac{2}{3}x_3 + \frac{4}{3}, x_3) \, : \, x_3 \in \boldsymbol{R}\}.$$

We have infinitely many solutions.

Observation. If we change the right hand side of the third equation in the last example as follows:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 3x_2 + 2x_3 &= 4 \\ 6x_2 + 4x_3 &= 1 \end{aligned}$$

we have a system that has no solutions.

It should be apparent that in all the examples we have seen the unknowns do not play any role in the solution of the problem, the coefficients being the important part in the process. We will study a system using only the coefficients without carrying the variables in our next example.

Solve the system

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= -2 \\ x_1 - x_2 + 3x_3 - x_4 &= 0 \\ x_2 + 2x_4 &= 0 \end{aligned}$$

We star by writing the coefficients only and in such a way that their placement regarding the unknowns they multiply remains clear in the process:

$$\begin{array}{cccccc} 2 & -1 & 0 & 0 & = & 1 \\ -1 & 2 & -1 & 0 & = & -2 \\ 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 0 & 2 & = & 0 \end{array}$$

We want to have a one on the place of the first entry of our array. We can achieve this either by multiplying the first equation by $\frac{1}{2}$ or by interchanging the firs and third equations. We do the later

$$\begin{array}{cccccc} 1 & -1 & 3 & -1 & = & 0 \\ -1 & 2 & -1 & 0 & = & -2 \\ 2 & -1 & 0 & 0 & = & 1 \\ 0 & 1 & 0 & 2 & = & 0 \end{array}$$

Now we want the 1 in the first place of the first equation tho be the only non zero entry in he first column. To get this, we add the first equation to the second and we add -2 times the first equation to the third

$$\begin{array}{rrrrcr} 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 2 & -1 & = & -2 \\ 0 & 1 & -6 & 2 & = & 1 \\ 0 & 1 & 0 & 2 & = & 0 \end{array}$$

The second equation has a 1 in the place that represents x_2. We want to have only zeros below this 1. We get this by subtracting the second equation from the third equation and the fourth equation:

$$\begin{array}{rrrrcr} 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 2 & -1 & = & -2 \\ 0 & 0 & -8 & 3 & = & 3 \\ 0 & 0 & -2 & 3 & = & 2 \end{array}$$

We want the third equation to start with 1 in the place of x_3. We will interchange the third and fourth equations:

$$\begin{array}{rrrrcr} 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 2 & -1 & = & -2 \\ 0 & 0 & -2 & 3 & = & 2 \\ 0 & 0 & -8 & 3 & = & 3 \end{array}$$

We multiply the third equation by $-\frac{1}{2}$:

$$\begin{array}{rrrrcr} 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 2 & -1 & = & -2 \\ 0 & 0 & 1 & -\frac{3}{2} & = & -1 \\ 0 & 0 & -8 & 3 & = & 3 \end{array}$$

We add 8 times the third equation to the fourth equation:

$$\begin{array}{rrrrcr} 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 2 & -1 & = & -2 \\ 0 & 0 & 1 & -\frac{3}{2} & = & -1 \\ 0 & 0 & 0 & -9 & = & -5 \end{array}$$

We multiply the last equation by $\frac{1}{9}$:

$$\begin{array}{cccccc} 1 & -1 & 3 & -1 & = & 0 \\ 0 & 1 & 2 & -1 & = & -2 \\ 0 & 0 & 1 & -\frac{3}{2} & = & -1 \\ 0 & 0 & 0 & 1 & = & \frac{5}{9} \end{array}$$

Now we backsolve

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0 \\ x_2 + 2x_3 - x_4 &= -2 \\ x_3 - \frac{3}{2}x_4 &= -1 \\ x_4 &= \frac{5}{9} \end{aligned}$$

We get:

$$\begin{aligned} x_3 - \frac{5}{6} &= -1 \\ x_3 &= -1 + \frac{5}{6} = -\frac{1}{6} \\ x_2 - \frac{1}{3} - \frac{5}{9} &= -2 \\ x_2 &= \frac{3 + 5 - 18}{9} = -\frac{10}{9} \\ x_1 + \frac{10}{9} - \frac{1}{6} - \frac{5}{9} &= 0 \\ x_1 &= \frac{-20 + 3 + 10}{18} = -\frac{7}{18} \end{aligned}$$

The solution set is:

$$S = \{(-\frac{7}{18}, -\frac{10}{9}, -\frac{1}{6}, \frac{5}{9})\}.$$

As you have seen, we placed the data from the system in a rectangular array of numbers and one column of equal signs. What we used are matrices and this will be the topic of our next session.