Lecture 7

Definition. Let $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{R}^n$, both nonzero vectors. We define the angle θ between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ to be the angle, measured in radians,such that $0 \le \theta \le \pi, \cos(\theta) = \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|}$.

Examples. The angle between the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given by:

$0 \le \theta \le \pi$

$$\cos(\theta) = \frac{1}{\sqrt{2}}.$$

Thus $\theta = \frac{\pi}{4}$.

The angle between the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is given by: $0 \le \theta \le \pi$

$$\cos(\theta) = \frac{-1}{1} = -1$$

$$\theta = \pi$$

The angle between the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is given by: $0 \le \theta \le \pi$

$$\cos(\theta) = \frac{-1}{\sqrt{2}}$$

$$\theta = \frac{3\pi}{4}.$$

Let $A = [a_{i,j}]_{\substack{i=1..n \\ j=1..k}}$ be an $n \times k$ matrix and let us write it by columns, so

$$A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_k].$$

Notice that if $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_k$ are the standard unit vectors in $\boldsymbol{R}^k$ then

$$Ae_1 = [a_{i,j}]_{\substack{i=1..n\\j=1..k}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ . \\ . \\ a_{n,1} \end{bmatrix} = \boldsymbol{c}_1$$

$$Ae_2 = \boldsymbol{c}_2$$

$$.$$

$$.$$

$$A\boldsymbol{e}_k = \boldsymbol{c}_k.$$

Thus we can write

$$A = [A\boldsymbol{e}_1, A\boldsymbol{e}_2, \ldots, A\boldsymbol{e}_k\ ].$$

This fact will become useful later on.

We also see that if $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are the standard unit vectors in $\boldsymbol{R}^n$ then

$$\boldsymbol{e}_1^t A = [a_{1,1}, a_{1,2}, \ldots, a_{1,k}] = \boldsymbol{r}_1$$

$$\boldsymbol{e}_2^t A = [a_{2,1}, a_{2,2}, \ldots, a_{2,k}] = \boldsymbol{r}_2$$

$$.$$

$$.$$

$$.$$

$$\boldsymbol{e}_n A = [a_{n,1} a_{n,2} \ldots, a_{n,k}] = \boldsymbol{r}_n$$

Hence we can also write

$$A = \begin{bmatrix} \boldsymbol{e}_1^t A \\ \boldsymbol{e}_2^t A \\ . \\ . \\ . \\ \boldsymbol{e}_n A \end{bmatrix}$$

Another useful fact is that if $\boldsymbol{x} \in \boldsymbol{R}^k, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_k \end{bmatrix}$ then

$$\boldsymbol{x} = \sum_{j=1}^{k} x_j \boldsymbol{e}_j$$

and so:.

$$\begin{aligned} A\boldsymbol{x} &= A(\sum_{j=1}^{k} x_j \boldsymbol{e}_j) \\ &= \sum_{j=1}^{k} x_j A \boldsymbol{e}_j \\ &= \sum_{j=1}^{k} x_j \boldsymbol{c}_j. \end{aligned}$$

Inverse Matrices

We will consider in this discussion square matrices of order n.

Proposition. Let A, B, C be square matrices of order n. If

$$BA = I_n$$
$$AC = I_n$$

then $B = C$.

Proof.
We have:

$$\begin{aligned} B &= BI_n \\ &= B(AC) \\ &= (BA)C \\ &= I_nC \\ &= C. \end{aligned}$$

Definition. Let $A = [a_{i,j}]_{\substack{i=1..n \\ j=1..n}}$ be a square matrix of order n We will say that A has an inverse if there exists a matrix B such that

$$AB = BA = I_n.$$

The proposition above tells us that if a matrix B exists with these properties then it is the only matrix that has these properties. We say that B is the inverse matrix of A and write

$$B = A^{-1}.$$

A useful fact that we will NOT prove is that if A is a square matrix of order n and if B is also a square matrix of order n such that $AB = I_n$ then also $BA = I_n$.

Example. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$. Then

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} + (-1)(-\frac{2}{3}) & \frac{1}{3} - \frac{1}{3} \\ \frac{2}{3} - \frac{2}{3} & \frac{2}{3} + \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2. \end{aligned}$$

We know (without proof) that then also $BA = I_2$ so we have that

$$A^{-1} = B.$$

Observation. Let A be a square matrix of order n and assume that A has an inverse matrix A^{-1}. If $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$ is any vector in $\boldsymbol{R}^n$ and if $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ is an arbitrary member of $\boldsymbol{R}^n$ then the equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

represents an arbitrary system of n linear equations in n unknowns that has A as he matrix of the coefficients.

Since A has an inverse, we have that

$$(A^{-1})(A\boldsymbol{x}) = A^{-1}\boldsymbol{b}$$
$$(A^{-1}A)\boldsymbol{x} = A^{-1}\boldsymbol{b}$$
$$\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

Thus the system has a unique solution for any $\boldsymbol{b}$.

We use this observation to show that $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$ cannot have an inverse. If we look at the system:

$$A\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

then the system is

$$x_1 + 2x_2 = 2$$
$$-3x_1 - 6x_2 = 3.$$

If we add three time the first equation to the second equation, we get:

$$x_1 + 2x_2 = 2$$
$$0 = 9.$$

The system has no solution so A cannot have an inverse.

Let us consider a two by two matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We would like to know when this matrix has an inverse and, when the inverse exists, to find it. Thus we want to see when it is possible to find real numbers $x_1, x_2 x_3, x_4$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} ax_1 + bx_3 &= 1 \\ ax_2 + bx_4 &= 0 \\ cx_1 + dx_3 &= 0 \\ cx_2 + dx_4 &= 1. \end{aligned}$$

If $a = 0$ then we need

$$\begin{aligned} bx_3 &= 1 \\ bx_4 &= 0 \\ cx_1 + dx_3 &= 0 \\ cx_2 + dx_4 &= 1 \end{aligned}$$

and we have immediately that $b \neq 0$ and

$$\begin{aligned} x_3 &= \frac{1}{b} \\ x_4 &= 0 \\ cx_1 + \frac{d}{b} &= 0 \\ cx_2 &= 1 \end{aligned}$$

If $c = 0$ there cannot be a solution, so $c \neq 0$ and

$$\begin{aligned} x_3 &= \frac{1}{b} \\ x_4 &= 0 \\ x_1 &= -\frac{d}{cb} \\ x_2 &= \frac{1}{c} \end{aligned}$$

We conclude that if

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

with

$$b \neq 0$$
$$c \neq 0$$

then A has an inverse and

$$A^{-1} = \begin{bmatrix} -\frac{d}{cb} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$$

It is easy to check that this is the inverse of A. Furthermore it is easy to see that *if*a^{-1} exists, then $b \neq 0, c \neq 0$.

If $a \neq 0$ then

$$ax_1 + bx_3 = 1$$
$$ax_2 + bx_4 = 0$$
$$cx_1 + dx_3 = 0$$
$$cx_2 + dx_4 = 1.$$

and we proceed with the information that $a \neq 0$, making it possible to multiply the first and second equations by $\frac{1}{a}$:

$$x_1 + \frac{b}{a}x_3 = \frac{1}{a}$$
$$x_2 + \frac{b}{a}x_4 = 0$$
$$cx_1 + dx_3 = 0$$
$$cx_2 + dx_4 = 1.$$

Now we cancel x_1 from the third equation and x_2 from that fourth equation:

$$x_1 + \frac{b}{a}x_3 = \frac{1}{a}$$
$$x_2 + \frac{b}{a}x_4 = 0$$
$$(d - \frac{cb}{a})x_3 = -\frac{c}{a}$$
$$(d - \frac{cb}{a})x_4 = 1.$$

The last equation has a solution if and only if

$$d - \frac{cb}{a} \neq 0$$
$$\frac{ad - cb}{a} \neq 0$$
$$ad - cb \neq 0.$$

Thus:

$$
\begin{aligned}
x_4 &= \frac{a}{ad - cb} \\
x_3 &= -\frac{c}{ad - cb} \\
x_1 + \frac{b}{a}x_3 &= \frac{1}{a} \\
x_1 - \frac{b}{a}\left(\frac{c}{ad - cb}\right) &= \frac{1}{a} \\
x_1 &= \frac{1}{a}\left(1 + \frac{bc}{ad - cb}\right) \\
&= \frac{ad - cb + cb}{a(ad - cb)} \\
&= \frac{d}{ad - cb} \\
x_2 + \frac{b}{a}x_4 &= 0 \\
x_2 &= -\frac{b}{a}\frac{a}{ad - cb} \\
&= -\frac{b}{ad - cb}.
\end{aligned}
$$

Thus A has an inverse if and only if $ad - cb \neq 0$ and if this is the case, we have

$$
A^{-1} = \begin{bmatrix} \frac{d}{ad-cb} & -\frac{b}{ad-cb} \\ -\frac{c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}.
$$

Definition. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We define the determinant of $A, \det(A) = ad - cb$.

Observation. We now can day that a square matrix of order 2 has an inverse if and only if $\det(A) \neq 0$.

Example. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Then $\det(A) = 8 + 3 = 11 \neq 0$. A has an inverse and

$$
A^{-1} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix}.
$$

To verify this we just take the product

$$
\begin{aligned}
AA^{-1} &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix} \\
&= \begin{bmatrix} \frac{8}{11} + \frac{3}{11} & \frac{2}{11} - \frac{2}{11} \\ \frac{12}{11} - \frac{12}{11} & \frac{3}{11} + \frac{8}{11} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$