Lecture 9

Now we will address deeper questions about matrices. The results we will see will allow us to determine, for example, given a system of linear equations which are the properties of the set of solutions, and, given a matrix of coefficients, how to find which systems with that matrix have solutions and the "structure" of the set of solutions to the equation.

$\boldsymbol{R}^n$ is an example of a vector space and we will deal with subsets of $\boldsymbol{R}^n$ which are vector spaces as well.

Definition. Let S be a nonempty subset of $\boldsymbol{R}^n$. We say that S is a subspace of $\boldsymbol{R}^n$ if:
1. For $\boldsymbol{x} \in S, \alpha \in \boldsymbol{R}, \alpha\boldsymbol{x} \in S$.
2. If $\boldsymbol{x},\boldsymbol{y} \in S$ then $\boldsymbol{x}+\boldsymbol{y} \in S$.

From the algebraic point of view, S has the same properties as $\boldsymbol{R}^n$.

Examples.
1. $\{0_n\}$ is a subspace of $\boldsymbol{R}^n$.

2. $\boldsymbol{R}^n$ is a subspace of $\boldsymbol{R}^n$.

Examples 1 and 2 are the called the trivial subspaces of $\boldsymbol{R}^n$.

3. Fix $1 \leq k \leq n$, and $S = \{\boldsymbol{x} = [x_1,x_2,\ldots,x_n] : x_i \in \boldsymbol{R}, x_k = 0\}$ is a subspace of $\boldsymbol{R}^n$.

4. Let $\boldsymbol{w}_0 \in \boldsymbol{R}^n$. Then $S = \{\gamma\boldsymbol{w}_0 : \gamma \in \boldsymbol{R}\}$ is a subspace of $\boldsymbol{R}^n$.

To see this, let $\boldsymbol{w} \in S, \alpha \in \boldsymbol{R}$. Then there exists $\gamma \in \boldsymbol{R}$ such that $\boldsymbol{w} = \gamma\boldsymbol{w}_0$ and so $\alpha\boldsymbol{w} = \alpha\gamma\boldsymbol{w}_0 \in S$.

If $\boldsymbol{w}_1,\boldsymbol{w}_2 \in S$, then there exist $\gamma_1,\gamma_2 \in \boldsymbol{R}$ such that $\boldsymbol{w}_1 = \gamma_1\boldsymbol{w}_0, \boldsymbol{w}_2 = \gamma_2\boldsymbol{w}_0$ and so $\boldsymbol{w}_1 + \boldsymbol{w}_2 = (\gamma_1+\gamma_2)\boldsymbol{w}_0 \in S$.

5. The only subspaces of $\boldsymbol{R}^2$ are: $\{0_2\}, \boldsymbol{R}^2$ and, for any $\boldsymbol{w}_0 \in \boldsymbol{R}^2, S = \{\gamma\boldsymbol{w}_0 : \gamma \in \boldsymbol{R}\}$.

To see this, suppose that there is a subspace S of $\boldsymbol{R}^2$ such that there is $\boldsymbol{w}_0 \neq 0, \boldsymbol{w}_0 \in S$ and suppose that there is $\boldsymbol{w}_1 \in S$ such that $\boldsymbol{w}_1 \neq \gamma\boldsymbol{w}_0$ for all $\gamma \in \boldsymbol{R}$.

Let us set $\boldsymbol{w}_0 = \begin{bmatrix} a \\ b \end{bmatrix}, \boldsymbol{w}_1 = \begin{bmatrix} c \\ d \end{bmatrix}$.

Note that $\boldsymbol{w}_1 \neq 0_2$.

Let $\boldsymbol{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \boldsymbol{R}^2$. The entries of $\boldsymbol{w}$ are then given real numbers.

We want to show that there exist real numbers x_1, x_2 such that

$$x_1\boldsymbol{w}_0 + x_2\boldsymbol{w}_1 = w.$$

If this is the case then, since $x_1\boldsymbol{w}_0 + x_2\boldsymbol{w}_1 \in S$ and $\boldsymbol{w}$ is arbitrary, we would have that $S = \boldsymbol{R}^2$, proving our claim.

By definition of equality we need to consider the system

$$ax_1 + cx_2 = x$$
$$bx_1 + dx_2 = y.$$

The matrix of the coefficients is $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and its determinant is $ad - cb$.

If $ad - cb \neq 0$ then the system has a solution (no matter what $\boldsymbol{w}$ is) and the conclusion is that $\boldsymbol{R}^2$ is a subset of S. Since $S \subseteq \boldsymbol{R}^2$ we then have that $S = \boldsymbol{R}^2$.

Assume that $\det(A) = 0$.

We have two cases (since not both a, b are zero):

Case 1. $a \neq 0$. Then

$$ad - cb = 0$$
$$d = \frac{cb}{a}.$$

Let $\gamma = \frac{c}{a}$. Then

$$\gamma\begin{bmatrix} a \\ b \end{bmatrix} = \frac{c}{a}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{ca}{a} \\ \frac{cb}{a} \end{bmatrix} = \begin{bmatrix} c \\ \frac{cb}{a} \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

This is a contradiction to the fact that $\boldsymbol{w}_1 \neq \gamma\boldsymbol{w}_0$.

Case 2. $b \neq 0$. Then

$$
\begin{aligned}
ad - cb &= 0 \\
c &= \frac{ad}{b}.
\end{aligned}
$$

Let $\gamma = \frac{d}{b}$ and we get:

$$
\begin{aligned}
\gamma \begin{bmatrix} a \\ b \end{bmatrix} &= \frac{d}{b} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} \frac{da}{b} \\ \frac{db}{b} \end{bmatrix} \\
&= \begin{bmatrix} c \\ d \end{bmatrix}.
\end{aligned}
$$

This is also a contradiction to the same fact.
Thus $\det(A) \neq 0$ and the conclusion is that $S = \boldsymbol{R}^2$.