THE SEMINORMAL REPRESENTATIONS OF THE BRAUER ALGEBRAS

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ABSTRACT. There will eventually be an abstract.

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1. INTRODUCTION

This paper will eventually have an introduction !

Editorial note: The main results of this paper are due to John Enyang. The contributions of the second author were largely editorial and expository.

At one point, we we intended to use the detailed description of the seminormal representations of the Brauer algebra obtained here as an aid to understanding Murphy type bases for the quotients of the Brauer algebras acting on tensor space. However, as it turned out, our approach in [1] did not require using the results of this paper.

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2. Preliminaries

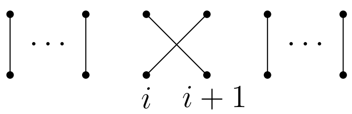
2.1. **Definition of the Brauer algebras.** The Brauer algebras were defined by Brauer [3]. Wenzl [18] showed that the Brauer algebras are obtained from the group algebra of the symmetric group by the Jones basic construction, and that the Brauer algebras over a field of characteristic zero are generically semisimple. Cellularity of the Brauer algebras was established by Graham and Lehrer [10].

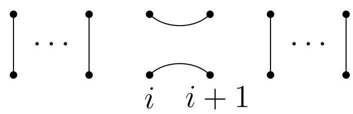
Let S be an integral domain with a distinguished element δ. The Brauer algebra $B_n = B_n(S;\delta)$ is the free S–module with basis the set of (n,n)–Brauer diagrams. The product of two Brauer diagrams is obtained by stacking them and then replacing each closed loop by a factor of δ; see [3] or [18] for details. By convention $B_0(S;\delta) = S$.

The generic ground ring for the Brauer algebras is $R = \mathbb{Z}[\boldsymbol{z}]$, where $\boldsymbol{z}$ is an indeterminant. We write $\mathbb{F}$ for the field of fractions $\mathbb{Q}(\boldsymbol{z})$ of R. For any specialization $B_k(S;\delta)$, one has $B_k(S;\delta) \cong B_k(R;\boldsymbol{z}) \otimes_R S$, with $\boldsymbol{z}$ acting on S by multiplication by δ, $\boldsymbol{z} \otimes 1 = 1 \otimes \delta$. We will commonly write $\mathcal{B}_k$ for $B_k(R;\boldsymbol{z})$ and $B_k(\boldsymbol{z})$ for $B_k(\mathbb{F};\boldsymbol{z})$.

The involution $*$ on (n,n)–Brauer diagrams which reflects a diagram in the axis $y = 1/2$ extends linearly to an algebra involution of $B_n(S;\delta)$. Note that the Brauer diagrams with only vertical strands are in bijection with permutations of $\{1,\dots,n\}$, and that the multiplication of two such diagrams coincides with the multiplication of permutations. Thus the Brauer algebra contains the group algebra $S\mathfrak{S}_n$ of the permutation group $\mathfrak{S}_n$ as a unital subalgebra. The identity element of the Brauer algebra is the diagram corresponding to the trivial permutation. We will note below that $S\mathfrak{S}_n$ is also a quotient of $B_n(S;\delta)$.

Let s_i and e_i denote the following (n,n)–Brauer diagrams:

$s_i =$ 

and $e_i =$ 

It is easy to see that $e_1,\dots,e_{n-1}$ and $s_1,\dots,s_{n-1}$ generate $B_n(S;\delta)$ as an algebra. We have $e_i^2 = \delta e_i$, so that e_i is an essential idempotent if $\delta \neq 0$ and nilpotent otherwise. Note that $e_i^* = e_i$ and $s_i^* = s_i$.

The products ab and ba of two Brauer diagrams have at most as many through strands as a. Consequently, the span of diagrams with fewer than n through strands is an ideal J_n in $B_n(S;\delta)$. The ideal J_n is generated by e_{n-1}. We have $B_n(S;\delta)/J_n \cong S\mathfrak{S}_n$, as algebras with involutions; in fact, the isomorphism is determined by $v + J_n \mapsto v$, for $v \in \mathfrak{S}_n$.

2.2. **Generic semisimplicity, branching diagram, and standard tableaux.** Let $\widehat{H}$ denote Young's lattice, i.e. the directed graded graph with vertices at level k the set $\widehat{H}_k$ of Young diagrams of size k, and directed edges $\lambda \to \mu$ connecting $\lambda \in \widehat{H}_{k-1}$ and $\mu \in \widehat{H}_k$ if μ is obtained from λ by adding one node. For $\mu \in \widehat{H}_k$, the set $A(\mu)$ of *addable nodes* of μ is the set of $\alpha = (j, \mu_j + 1)$ such that $\mu \cup \{\alpha\}$ is a Young diagram. Likewise, the set $R(\mu)$ of *removable nodes* is the set of $\alpha = (j, \mu_j)$ such that $\mu \setminus \{\alpha\}$ is a Young diagram. The addable nodes correspond one–to–one with $\lambda \in \widehat{H}_{k+1}$ such that $\mu \to \lambda$ in $\widehat{H}$, and the removable nodes correspond one–to–one with $\nu \in \widehat{H}_{k-1}$ such that $\nu \to \mu$ in $\widehat{H}$.

If K is a field of characteristic zero, then the group algebra $K\mathfrak{S}_k$ is split semisimple for all $k \geqslant 0$, and $\widehat{H}$ is the branching diagram for the tower of split semisimple algebras $(K\mathfrak{S}_k)_{k\geqslant 0}$.

It is well known that standard Young tableaux of a given shape $\lambda \in \widehat{H}_k$ can be identified with paths on $\widehat{H}$ from $\emptyset$ to λ, i.e. sequences $(\lambda^{(j)})_{0\leqslant j\leqslant k}$ such that $\lambda^{(j)} \in \widehat{H}_j$, $\lambda^{(k)} = \lambda$, and $\lambda^{(j-1)} \to \lambda^{(j)}$ in $\widehat{H}$ for $1 \leqslant j \leqslant k$.

We are going to define Brauer algebra analogues of Young's lattice and standard tableaux.

For $k \geqslant 0$, let $\widehat{B}_k$ be the set of pairs (λ, l), where $0 \leqslant l \leqslant \lfloor k/2 \rfloor$ and λ is Young diagram of size $k - 2l$. $\widehat{B}_0$ has a unique element $(\emptyset, 0)$, which we also denote by $\emptyset$. Define $\widehat{B}$ to be the directed graded graph with:

(1) vertices at level k: $\widehat{B}_k$, and
(2) a directed edge $(\lambda, l) \to (\mu, m)$ connecting $(\lambda, l) \in \widehat{B}_{k-1}$ and $(\mu, m) \in \widehat{B}_k$, if r μ is obtained either by adding a node to λ, or by deleting a node from λ.

According to Wenzl [18], Theorem 3.21 and Corollary 3.3, if K is a field of characteristic zero and $\delta \in K$ is not integer, then the algebras $B_k = B_k(K; \delta)$ are split semisimple, and the branching diagram for the sequence $(B_k)_{k \geqslant 0}$ is $\widehat{B}$. This applies in particular to $B_k(\boldsymbol{z}) = B_k(\mathbb{F}, \boldsymbol{z})$.

Next we discuss the analogue of standard tableaux.

Definition 2.1. A path on $\widehat{B}$ from $(\lambda, l) \in \widehat{B}_j$ to $(\mu, m) \in \widehat{B}_k$ (for $j < k$) is a sequence $\mathfrak{t} = (\mathfrak{t}^{(j)}, \mathfrak{t}^{(j+1)}, \dots, \mathfrak{t}^{(k)})$ with $\mathfrak{t}^{(j)} = (\lambda, l)$, $\mathfrak{t}^{(k)} = (\mu, m)$, $\mathfrak{t}^{(s)} \in \widehat{B}_s$ for all s, and $\mathfrak{t}^{(s)} \to \mathfrak{t}^{(s+1)}$ in $\widehat{B}$ for all $s < k$. The set of all paths from $\emptyset$ to $(\lambda, l) \in \widehat{B}_n$ is denoted by $\widehat{B}_n^{(\lambda, l)}$. If $\mathfrak{t} \in \widehat{B}_n^{(\lambda, l)}$, we write $\mathrm{Shape}(\mathfrak{t}) = (\lambda, l)$. Write $\widehat{B}_n^{(\cdot)}$ for the set of all paths from $\emptyset$ to some $(\lambda, l) \in \widehat{B}_n$.

The set $\widehat{B}_n^{((\lambda, l))}$ is the Brauer algebra analogue of the set of standard tableaux of fixed shape in the representation theory of the symmetric group. The set $\widehat{B}_n^{(\cdot)}$ is the Brauer algebra analogue of the set of all standard tableaux of fixed size n. These paths are often called "up–down tableaux".

Remark 2.2. For a path $\mathfrak{t} = \big((\lambda^{(0)}, l_0), (\lambda^{(1)}, l_1), \dots, (\lambda^{(k)}, l_k)\big) \in \widehat{B}_k^{(\lambda, l)}$, the sequence $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$ suffices to reconstruct $\mathfrak{t}$, since $l_j = (j - |\lambda^{(j)}|)/2$. Therefore, we will sometimes also write $\mathfrak{t}(j) = \lambda^{(j)}$.

Notation 2.3 (Operations on paths)**.**

- (Concatentation) A path $\mathfrak{s}$ from (λ, l) to (μ, m) and a path $\mathfrak{t}$ from (μ, m) to (ν, n) can be concatenated in the obvious sense; we denote the concatenation by $\mathfrak{s} \circ \mathfrak{t}$.
- (Truncation) If $\mathfrak{t}$ is a path from $\emptyset$ to $(\lambda, l) \in \widehat{B}_n$, and $0 \leqslant k < \ell \leqslant n$, write $t_{[k,\ell]}$ for the path $(\mathfrak{t}^{(k)}, \dots, \mathfrak{t}^{(\ell)})$. Write $\mathfrak{t}_{\downarrow j}$ for $\mathfrak{t}_{[0,j]}$. Write $\mathfrak{t}'$ for $\mathfrak{t}_{\downarrow n-1}$ and $\mathfrak{t}''$ for $\mathfrak{t}_{\downarrow n-2}$.
- (Shifting) If $\mathfrak{t} = ((\lambda^{(j)}, l_j))_{0 \leqslant j \leqslant k}$ is a path on $\widehat{B}$, let $\mathfrak{t}[2l] = ((\lambda^{(j)}, l_j + l))_{0 \leqslant j \leqslant k}$.

2.3. **Markov trace and conditional expectation.** Let $B_k = B_k(S; \delta)$ be any specialization of the Brauer algebras, where δ is assumed to be invertible in S. There exists a conditional expectation $\varepsilon_{k-1} : B_k \to B_{k-1}$, that is, a unital B_{k-1}–B_{k-1} bimodule map, defined on k–strand Brauer diagrams by $\varepsilon_{k-1}(b) = (1/\delta)\,\mathrm{cl}(b)$, where $\mathrm{cl}(b)$ is obtained by joining the k–th upper and lower vertices of b, and replacing any closed loop by a factor of δ. For $x \in B_k$, one has

$$e_k x e_k = \delta \varepsilon_{k-1}(x) e_k. \tag{2.1}$$

Lemma 2.4. *The map $x \mapsto x e_k$ from B_k to B_{k+1} is injective.*

Proof. One has $\varepsilon_k(x e_k) = (1/\delta) x$. $\square$

There exists an S–valued trace τ on B_k defined inductively by $\tau(1) = 1$ and $\tau(b) = \tau(\varepsilon_{k-1}(b))$ for $b \in B_k$. By definition $\tau(\iota(b)) = \tau(b)$ where ι denotes the embedding of B_k in B_{k+1}, and $\tau \circ \varepsilon_{k-1} = \tau$. The trace τ is called the Markov trace. See [18], Proposition 2.2 for details.

According to Wenzl [18], if K is a field of characteristic zero and $\delta \in K$ is not an integer, then the Markov trace τ is non–degenerate on $B_k = B_k(K; \delta)$; that is, for each $x \in B_k$, there exists $y \in B_k$ such that $\tau(xy) \neq 0$. This entails that $\tau(p) \neq 0$ for each minimal idempotent $p \in B_k$.

We conclude this subsection with some observations about minimal idempotents. Assume K is a field of characteristic zero and $\delta \in K$ is not an integer, as above. For $(\lambda, l) \in \widehat{B}_k$, let $z_{(\lambda,l)}$ denote the corresponding minimal central idempotent in $B_k = B_k(K;\delta)$.

Lemma 2.5. *Let $(\lambda, l) \in \widehat{B}_{k-1}$. If p is a minimal idempotent in $B_{k-1}z_{(\lambda,l)}$, then $(1/\delta)e_k p$ is a minimal idempotent in $B_{k+1}z_{(\lambda,l+1)}$*

Proof. By the proof of Theorem 3.2 in [18], $J_{k+1} = B_{k+1}e_kB_{k+1}$ is isomorphic to the Jones basic construction for the pair $B_{k-1} \subseteq B_k$. The result now follows from [18], Proposition 1.2. □

Lemma 2.6. *Let p be a minimal idempotent in B_{k-1} and ζ a minimal central idempotent in B_k such that $p\,\zeta \neq 0$. Then $\varepsilon_{k-1}(p\,\zeta) = (\tau(p\,\zeta)/\tau(p))\,p$.*

Proof. We have $\varepsilon_{k-1}(p\,\zeta) = p\varepsilon_{k-1}(p\zeta)p$, by the bimodule property of ε_{k-1}, and since p is a minimal idempotent, this is equal to αp for some $\alpha \in K$. Applying τ, we get $\tau(p\,\zeta) = \alpha\tau(p)$, so $\alpha = \tau(p\,\zeta)/\tau(p)$. □

2.4. **Weights of the Markov trace.** Wenzl has determined the weights of the Markov trace τ on $B_k(\boldsymbol{z}) = B_k(\mathbb{F}, \boldsymbol{z})$, that is, the values of τ on minimal idempotents.

Notation 2.7. For a Young diagram μ, let $\tilde{\mu}$ denote the transposed diagram. Thus $\tilde{\mu}_j$ is the length of the j–th column of μ. If (i, j) is a node of the Young diagram μ, the associated hook length is

$$h^{\mu}_{(i,j)} = \mu_i - i + \tilde{\mu}_j - j + 1.$$

Let $H(\mu)$ denote the produce of all the hook lengths, $H(\mu) = \prod_{\alpha\in\mu} h^{\mu}(\alpha)$, where the product is over all nodes α of μ.

Definition 2.8 ([4])**.** The El Samra–King polynomial associated to a Young diagram μ is

$$P_\mu(\boldsymbol{z}) = (1/H(\mu)) \prod_{\substack{(i,j)\in\mu\\ i\geqslant j}} (\boldsymbol{z} + \mu_i + \mu_j - i - j) \prod_{\substack{(i,j)\in\mu\\ i<j}} (\boldsymbol{z} - \tilde{\mu}_i - \tilde{\mu}_j + i + j - 2)$$

Theorem 2.9 ([18], Theorem 3.2)**.** *Let p be a minimal idempotent in the minimal ideal of $B_k(\boldsymbol{z})$ labelled by $(\mu, l) \in \widehat{B}_k$. Then*

$$\tau(p) = P_\mu(\boldsymbol{z})/\boldsymbol{z}^k.$$

2.5. **A Murphy basis for the Brauer algebras.** Introduced by Graham and Lehrer as a device for studying non–semisimple representations of a class of algebras that includes Hecke algebras, Schur algebras and Brauer algebras [10], cellular algebras are defined by the existence of a *cellular basis* and a *cell datum*, which have combinatorial properties analogous to the "Robinson–Schensted correspondence" in the group algebra of the symmetric group. Cellularity of $\mathcal{B}_k$ was established using a tangle type basis in [10, Theorem 3.11]. This paper will use the cellular basis for $\mathcal{B}_k$ given in [6, Section 6], which is a Brauer algebra analogue of Murphy's cellular basis [15, Theorem 4.17] for the Iwahori-Hecke algebra of the symmetric group. We will adhere closely to the notation established in [6]. For further details on cellular algebras in general, the reader is referred to [10, 12, 13, 8, 9].

For $i = 0, 1, \dots,$ let

$$e^{(l)}_{i-1} = \begin{cases} 1, & \text{if } l = 0 \text{ and } i \leqslant 2, \\ \underbrace{e_{i-2l+1}e_{i-2l+3}\cdots e_{i-1}}_{l \text{ factors}} & \text{if } 1 \leqslant l \leqslant \lfloor i/2 \rfloor, \\ 0, & \text{if } \lfloor i/2 \rfloor < l. \end{cases}$$

For $(\lambda, l) \in \widehat{B}_k$, define

$$c_{(\lambda,0)} = \sum_{w \in \mathfrak{S}_\lambda} w \qquad \text{and} \qquad c_{(\lambda,l)} = c_{(\lambda,0)} e^{(l)}_{k-1}.$$

If $1 \leqslant a < i$, define

$$w_{a,i} = s_a s_{a+1} \cdots s_{i-1} = (i, i-1, \dots, a)$$

and $w_{i,a} = w^*_{a,i}$. If $\mu \vdash i-1$ and $\lambda \vdash i$, with $\lambda = \mu \cup \{(j, \lambda_j)\}$, let $a = \sum_{r=1}^{j} \lambda_r$ and define

$$d^{(i)}_{\mu\to\lambda} = w_{a,i} \qquad \text{and} \qquad u^{(i)}_{\mu\to\lambda} = w_{i,a} \sum_{r=0}^{\mu_j} w_{a,a-r}.$$

The elements $d^{(i)}_{\mu\to\lambda}$ and $u^{(i)}_{\mu\to\lambda}$ are "branching factors" for restriction and induction of cell modules of the symmetric group algebras $\mathbb{Z}\mathfrak{S}_n$. They are related to the Murphy basis of the symmetric group algebras as follows. For $n \geqslant 0$ and $\lambda \in \widehat{H}_n$, identify a standard tableau $\mathfrak{t}$ of shape λ with a path $(\lambda^{(j)})_{0\leqslant j\leqslant n}$ on Young's lattice $\widehat{H}$ and define

$$d_{\mathfrak{t}} = d^{(n)}_{\lambda^{(n-1)}\to\lambda^{(n)}} d^{(n-1)}_{\lambda^{(n-2)}\to\lambda^{(n-1)}} \cdots d^{(1)}_{\lambda^{(0)}\to\lambda^{(1)}}.$$

Then $d_{\mathfrak{t}} \in \mathfrak{S}_n$ is the permutation such that $\mathfrak{t}^\lambda d_{\mathfrak{t}} = \mathfrak{t}$, where $\mathfrak{t}^\lambda$ is the "row reading" tableau in which the entries $1, \dots n$ are entered in order along the rows. Murphy's basis is recovered as

$$m_{\mathfrak{s}\,\mathfrak{t}} = d^*_{\mathfrak{s}} c_{(\lambda,0)} d_{\mathfrak{t}}.$$

See [6, Section 4].

For future reference we recall from [6, Lemma A.2] that the d– and u– branching factors for the symmetric groups satisfy the compatibility relation

$$c_{(\lambda,0)} d^{(i)}_{\mu\to\lambda} = (u^{(i)}_{\mu\to\lambda})^* c_{(\mu,0)}. \tag{2.2}$$

It is shown in [6] that the branching factors for the symmetric group algebras can be lifted to branching factors for the Brauer algebras, and that an analogue of Murphy's cellular basis is obtained by taking ordered products of d– branching factors along paths on $\widehat{B}$, as follows:

For $(\mu, m) \in \widehat{B}_{k-1}$ and $(\lambda, l) \in \widehat{B}_k$, with $(\mu, m) \to (\lambda, l)$ in $\widehat{B}$, define

$$d^{(k)}_{(\mu,m)\to(\lambda,l)} = \begin{cases} d^{(k-2m)}_{\mu\to\lambda} e^{(m)}_{k-2}, & \text{if } l = m, \\ u^{(k-2m-1)}_{\lambda\to\mu} e^{(m)}_{k-2}, & \text{if } l = m+1. \end{cases} \tag{2.3}$$

If $(\mu, m) \in \widehat{B}_k$ and $\mathfrak{t} = ((\mu^{(0)}, m_0), \dots, (\mu^{(k)}, m_k)) \in \widehat{B}^{(\mu,m)}_k$, define

$$d_{\mathfrak{t}} = d^{(k)}_{(\mu^{(k-1)},m_{k-1})\to(\mu^{(k)},m_k)} d^{(k-1)}_{(\mu^{(k-2)},m_{k-2})\to(\mu^{(k-1)},m_{k-1})} \cdots d^{(1)}_{(\mu^{(0)},m_0)\to(\mu^{(1)},m_1)}.$$

We now extend the usual dominance order $\trianglerighteq$ on partitions to an order on the set $\widehat{B}_k$.

Definition 2.10. Let $(\lambda, l), (\mu, m) \in \widehat{B}_k$. Write $(\lambda, l) \trianglerighteq (\mu, m)$ if either $l > m$, or $l = m$ and $\lambda \trianglerighteq \mu$.

Proposition 2.11 ([6, Theorem 6.19]). *The set*

$$\mathscr{B}_k = \{m_{\mathfrak{s}\,\mathfrak{t}} = d_{\mathfrak{s}}^* c_{(\lambda,l)} d_{\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)},\ (\lambda,l) \in \widehat{B}_k\} \tag{2.4}$$

is an R-basis for $\mathcal{B}_k$ and $(\mathcal{B}_k, *, \widehat{B}_k, \trianglerighteq, \mathscr{B}_k)$ is a cell datum for $\mathcal{B}_k$.

One also has u–branching factors for the Brauer algebras defined as follows: For $(\mu,m) \in \widehat{B}_{k-1}$ and $(\lambda,l) \in \widehat{B}_k$, with $(\mu,m) \to (\lambda,l)$ in $\widehat{B}$, define

$$u^{(k)}_{(\mu,m)\to(\lambda,l)} = \begin{cases} u^{(k-2m)}_{\mu\to\lambda} e^{(m)}_{k-1}, & \text{if } l = m, \\ d^{(k-2m-1)}_{\lambda\to\mu} e^{(m+1)}_{k-1}, & \text{if } l = m+1. \end{cases} \tag{2.5}$$

The d– and u–branching factors satisfy the following compatibility relation:

$$c_{(\lambda,l)} d^{(k)}_{(\mu,m)\to(\lambda,l)} = (u^{(k)}_{(\mu,m)\to(\lambda,l)})^* c_{(\mu,m)}, \tag{2.6}$$

see [6, Appendix A]. This relation will be used frequently.

We recall for convenience some basic properties of the bases (2.4) derived from the general theory of cellular algebras (for details, see [10, 9, 8, 6]). For $(\lambda,l) \in \widehat{B}_k$, let $\mathcal{B}_k^{\trianglerighteq(\lambda,l)}$ denote the R–span of $\{m_{\mathfrak{s}\,\mathfrak{t}} \mid (\mu,m) \trianglerighteq (\lambda,l)$ and $\mathfrak{s},\mathfrak{t} \in \widehat{B}_k^{(\mu,m)}\}$ and $\mathcal{B}_k^{\triangleright(\lambda,l)}$ the R–span of $\{m_{\mathfrak{s}\,\mathfrak{t}} \mid (\mu,m) \triangleright (\lambda,l)$ and $\mathfrak{s},\mathfrak{t} \in \widehat{B}_k^{(\mu,m)}\}$. Then these are two sided ideals in $\mathcal{B}$. Indeed, one has

$$\mathcal{B}_k^{\trianglerighteq(\lambda,l)} = \sum_{(\mu,m)\trianglerighteq(\lambda,l)} \mathcal{B}_k c_{(\mu,m)} \mathcal{B}_k,$$

and similarly for $\mathcal{B}_k^{\triangleright(\lambda,l)}$. For $(\lambda,l) \in \widehat{B}_k$ and $\mathfrak{s},\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$, and $b \in \mathcal{B}$, one has

$$m_{\mathfrak{s}\,\mathfrak{t}} b \equiv \sum_{\mathfrak{u}} r_{\mathfrak{u}}(b;t) m_{\mathfrak{s}\,\mathfrak{u}} \mod \mathcal{B}_k^{\triangleright(\lambda,l)}, \tag{2.7}$$

where the sum is over $\mathfrak{u} \in \widehat{B}_k^{(\lambda,l)}$, and the scalars $r_{\mathfrak{u}}(b;t) \in R$ depend only on b and $\mathfrak{t}$, and not on $\mathfrak{s}$. Evidently, one has $m_{\mathfrak{s}\,\mathfrak{t}}^* = m_{\mathfrak{t}\,\mathfrak{s}}$, and it follows that

$$b m_{\mathfrak{s}\,\mathfrak{t}} \equiv \sum_{\mathfrak{u}} r_{\mathfrak{u}}(b^*;s) m_{\mathfrak{u}\,\mathfrak{t}} \mod \mathcal{B}_k^{\triangleright(\lambda,l)}. \tag{2.8}$$

The *cell module* $\Delta_k^{(\lambda,l)}$ is the free R-module with basis $\{m_{\mathfrak{t}} \mid \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}\}$ with right $\mathcal{B}_k$-action

$$m_{\mathfrak{t}} b = \sum_{\mathfrak{u}} r_{\mathfrak{u}}(b;t) m_{\mathfrak{u}}, \quad \text{for } b \in \mathcal{B}_k,$$

where the scalars $r_{\mathfrak{u}}(b;t) \in R$ are determined by the relation (2.7). The *opposite cell module* $(\Delta_k^{(\lambda,l)})^*$ is the free R–module with basis $\{m_{\mathfrak{s}}^* \mid \mathfrak{s} \in \widehat{B}_k^{(\lambda,l)}\}$ and left action

$$b m_{\mathfrak{s}} = \sum_{\mathfrak{u}} r_{\mathfrak{u}}(b^*;s) m_{\mathfrak{u}}^*, \quad \text{for } b \in \mathcal{B}_k.$$

It follows from the definitions that the map $\alpha_{(\lambda,l)} : \mathcal{B}_k^{\trianglerighteq(\lambda,l)}/\mathcal{B}_k^{\triangleright(\lambda,l)} \to (\Delta_k^{(\lambda,l)})^* \otimes_R \Delta_k^{(\lambda,l)}$ determined by $m_{\mathfrak{s}\,\mathfrak{t}} + \mathcal{B}_k^{\triangleright(\lambda,l)} \mapsto m_s^* \otimes m_t$, for $\mathfrak{s},\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$, is a $\mathcal{B}_k$–$\mathcal{B}_k$ bimodule isomorphism.

For $(\lambda,l) \in \widehat{B}_k$ and $\mathfrak{s},\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$, the map $\langle\,,\rangle : \Delta_k^{(\lambda,l)} \times \Delta_k^{(\lambda,l)} \to R$, defined by

$$\langle m_{\mathfrak{u}}, m_{\mathfrak{v}} \rangle m_{\mathfrak{s}\,\mathfrak{t}} \equiv m_{\mathfrak{s}\,\mathfrak{u}} m_{\mathfrak{v}\,\mathfrak{t}} \mod \mathcal{B}_k^{\triangleright(\lambda,l)} \qquad \text{for } \mathfrak{u},\mathfrak{v} \in \widehat{B}_k^{(\lambda,l)}, \tag{2.9}$$

is a symmetric bilinear form on $\Delta_k^{(\lambda,l)}$. It is shown in [10, Sect. 2] that the form defined by the relation (2.9) is independent of the choice of $\mathfrak{s}$ and $\mathfrak{t}$. It does depend, however, on the choice of the cellular basis $\mathscr{B}_k = \{m_{\mathfrak{s}\,\mathfrak{t}}\}$ of $\mathcal{B}_k$.

The basis for $\Delta_k^{(\lambda,l)}$ realises an explicit filtration by cell modules for $\mathcal{B}_{k-1}$ of the restriction of $\Delta_k^{(\lambda,l)}$ to $\mathcal{B}_{k-1}$.

Proposition 2.12 ([6, Lemma 3.13]). *Assume that $(\lambda, l) \in \widehat{B}_k$ and $(\rho, r) \in \widehat{B}_{k-1}$, where $(\rho, r) \to (\lambda, l)$ in $\widehat{B}$. Let $N^{\trianglerighteq(\rho,r)} \subseteq \Delta_k^{(\lambda,l)}$ and $N^{\triangleright(\rho,r)} \subseteq \Delta_k^{(\lambda,l)}$ denote the R-submodules respectively generated by*

$$\{m_\mathfrak{t} \in \Delta_k^{(\lambda,l)} \mid \mathrm{Shape}(\mathfrak{t}_{\downarrow k-1}) \trianglerighteq (\rho, r)\} \qquad \textit{and} \qquad \{m_\mathfrak{t} \in \Delta_k^{(\lambda,l)} \mid \mathrm{Shape}(\mathfrak{t}_{\downarrow k-1}) \triangleright (\rho, r)\}.$$

Then the linear map $N^{\trianglerighteq(\rho,r)}/N^{\triangleright(\rho,r)} \to \Delta_{k-1}^{(\rho,r)}$ given by

$$m_\mathfrak{s} + N^{\triangleright(\rho,r)} \mapsto m_\mathfrak{t}, \qquad \textit{if } \mathfrak{s} \in \widehat{B}_k^{(\lambda,l)},\ \mathfrak{t} \in \widehat{B}_{k-1}^{(\rho,r)} \textit{ and } \mathfrak{s}_{\downarrow k-1} = \mathfrak{t},$$

is an isomorphism of $\mathcal{B}_{k-1}$-modules.

2.6. **A partial order on paths, and a maximal path.** The following partial order on paths will play a crucial role:

Definition 2.13 ([9, Definition 2.16]). Consider paths $\mathfrak{s} = ((\lambda^{(0)}, l_0), \dots, (\lambda^{(k)}, l_k))$ and $\mathfrak{t} = ((\mu^{(0)}, m_0), \dots, (\mu^{(k)}, m_k))$ in $\widehat{B}_k^{(\cdot)}$. We say that $\mathfrak{s}$ *precedes* $\mathfrak{t}$ in *reverse lexicographic order* (denoted $\mathfrak{t} \succcurlyeq \mathfrak{s}$) if $\mathfrak{s} = \mathfrak{t}$, or if for the last index i such that $(\lambda^{(i)}, l_i) \neq (\mu^{(i)}, m_i)$ we have $(\mu^{(i)}, m_i) \triangleright (\lambda^{(i)}, l_i)$. Write $\mathfrak{t} \succ \mathfrak{s}$ if $\mathfrak{t} \succcurlyeq \mathfrak{s}$ and $\mathfrak{s} \neq \mathfrak{t}$.

Next, we define two distinguished paths in $\widehat{B}_k^{(\lambda,l)}$. Recall the "row reading" standard tableau $\mathfrak{t}^\lambda$ for $\lambda \in \widehat{H}_n$, namely the standard tableau in which the numbers $1, \dots, n$ are entered in order from left to right along the rows of λ. Regarded as a path on $\widehat{H}$, $\mathfrak{t}^\lambda = (\lambda^{(j)})_{0 \leqslant j \leqslant n}$, where $\lambda^{(j)} \setminus \lambda^{(j-1)}$ is the node in $A(\lambda^{(j-1)}) \cap \lambda$ with least row index.

Definition 2.14.
(1) For $(\lambda, l) \in \widehat{B}_k$, define $\mathfrak{t} = \mathfrak{t}^{(\lambda,l)}$ by

$$\begin{aligned} \mathfrak{t}^{(2r)} &= (\emptyset, r) && \text{for} \quad 0 \leqslant r \leqslant l \\ \mathfrak{t}^{(2r+1)} &= ((1), r) && \text{for} \quad 0 \leqslant r \leqslant l-1, \end{aligned}$$

and $\mathfrak{t}_{[2l,k]} = ((\lambda^{(j)}, l))_{0 \leqslant j \leqslant k-2l}$, where $\mathfrak{t}^\lambda = (\lambda^{(j)})_{0 \leqslant j \leqslant k-2l}$.
(2) Let $(\lambda, l) \in \widehat{B}_k$. Let ρ be the Young diagram obtained by adjoining to λ its lowest addable node, $\rho = \lambda \cup \{(\ell+1, 1)\}$, where ℓ is the length of λ. Define $\mathfrak{u} = \mathfrak{u}^{(\lambda,l)} \in \widehat{B}_k^{(\lambda,l)}$ by $\mathfrak{u}_{\downarrow k-2l} = \mathfrak{t}^{(\lambda,0)}$, and

$$\begin{aligned} \mathfrak{u}^{(k-2l+2r)} &= (\lambda, r) && \text{for} \quad 0 \leqslant r \leqslant l \\ \mathfrak{u}^{(k-2l+2r+1)} &= (\rho, r) && \text{for} \quad 0 \leqslant r \leqslant l-1, \end{aligned}$$

Remark 2.15. For $(\lambda, l) \in \widehat{B}_k$, $\mathfrak{t}^{(\lambda,l)}$ is characterized as the unique maximal element in $\widehat{B}_k^{(\lambda,l)}$ with respect to reverse lexicographic order. Note also that $\mathfrak{t}^{(\emptyset,l)} = \mathfrak{u}^{(\emptyset,l)}$ and $\mathfrak{t}^{(\lambda,l)} = \mathfrak{t}^{(\emptyset,l)} \circ \mathfrak{t}^{(\lambda,0)}[2l]$.

Notation 2.16. For $x \in \mathcal{B}_j$ and $r \geqslant 1$, define $x[r] \in \mathcal{B}_{j+r}$ to be x shifted by r strands to the right. For the generators e_i, s_i, we have $e_i[r] = e_{i+r}$ and $s_i[r] = s_{i+r}$.

The following consequence of the contraction identity for the e_i's will be used frequently.

Lemma 2.17. *For $l \geqslant 1$ and $k \geqslant 3$,*

$$e_k^{(l)} e_{k-1}^{(l-1)} e_{k-2}^{(l-1)} = e_k^{(l)}.$$

Proof. This is evident if $l = 1$. For $l > 1$,

$$e_k^{(l)} e_{k-1}^{(l-1)} e_{k-2}^{(l-1)} = e_k e_{k-2}^{(l-1)} e_{k-1}^{(l-1)} e_{k-2}^{(l-1)} = e_k^{(l)}.$$

$\square$

Lemma 2.18. *Let $k \geqslant 1$, $(\lambda, l) \in \widehat{B}_k$, and $\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$. Then the following statements hold:*

(1) $c_{(\lambda,l)} d_{\mathfrak{u}^{(\lambda,l)}} = c_{(\lambda,l)}$.
(2) $d_{\mathfrak{t}^{(\lambda,0)}} = 1$ and, for $l \geqslant 1$,

$$d_{\mathfrak{t}^{(\lambda,l)}} = \begin{cases} e_{k-2}^{(l)} e_{k-3}^{(l)} \cdots e_{2l-1}^{(l)}, & \textit{if } \lambda \neq \emptyset \textit{ and } k > 2, \\ e_{k-2}^{(l-1)} e_{k-3}^{(l-1)}, & \textit{if } \lambda = \emptyset \textit{ and } k > 2, \\ 1 & \textit{if } k = 2. \end{cases}$$

(3) For $l \geqslant 1$ and $x \in \mathcal{B}_{k-2l}$,

$$x e_{k-1}^{(l)} e_{k-2}^{(l)} \cdots e_{2l-1}^{(l)} = e_{k-1}^{(l)} e_{k-2}^{(l)} \cdots e_{2l-1}^{(l)} x[2l].$$

Proof. (1) Let $\mathfrak{u} = \mathfrak{u}^{(\lambda,l)}$. If $l = 0$, then $\mathfrak{u} = \mathfrak{t}^{(\lambda,0)}$, and $d_{\mathfrak{u}} = 1$. So assume $l > 0$, and proceed by induction on l. Write $\mathfrak{u}'' = \mathfrak{u}_{\downarrow k-2}$.

Note that $d_{\lambda\to\rho}^{(k-2l+1)} = u_{\lambda\to\rho}^{(k-2l+1)} = 1$. Therefore $d_{\mathfrak{u}^{(k-1)}\to\mathfrak{u}^{(k)}}^{(k)} = d_{(\rho,l-1)\to(\lambda,l)}^{(k)} = e_{k-2}^{(l-1)}$. Likewise, $d_{\mathfrak{u}^{(k-2)}\to\mathfrak{u}^{(k-1)}}^{(k-1)} = d_{(\lambda,l-1)\to(\rho,l-1)}^{(k-1)} = e_{k-3}^{(l-1)}$. Hence,

$$c_{(\lambda,l)} d_{\mathfrak{u}^{(k-1)}\to\mathfrak{u}^{(k)}}^{(k)} d_{\mathfrak{u}^{(k-2)}\to\mathfrak{u}^{(k-1)}}^{(k-1)} = c_{(\lambda,0)} e_{k-1}^{(l)} e_{k-2}^{(l-1)} e_{k-3}^{(l-1)} = c_{(\lambda,0)} e_{k-1}^{(l)} = c_{(\lambda,l)},$$

using Lemma 2.17. Thus, writing $\mathfrak{u}'' = \mathfrak{u}_{\downarrow k-2}$,

$$c_{(\lambda,l)} d_{\mathfrak{u}} = c_{(\lambda,l)} d_{\mathfrak{u}''}.$$

The result follows from the induction hypothesis, since $c_{(\lambda,l)} = e_{k-1} c_{(\lambda,l-1)}$ and $\mathfrak{u}'' = \mathfrak{u}^{(\lambda,l-1)}$.

(2) Compute directly that $d_{\mathfrak{t}^{(\emptyset,1)}} = 1$ and $d_{\mathfrak{t}^{(\emptyset,2)}} = e_2 e_1$. For $l \geqslant 3$,

$$d_{\mathfrak{t}^{(\emptyset,l)}} = e_{2l-2}^{(l-1)} e_{2l-3}^{(l-1)} d_{\mathfrak{t}^{(\emptyset,l-1)}} = e_{2l-2}^{(l-1)} e_{2l-3}^{(l-1)} e_{2l-4}^{(l-2)} e_{2l-5}^{(l-2)} = e_{2l-2}^{(l-1)} e_{2l-3}^{(l-1)}.$$

by the appropriate induction hypothesis and Lemma 2.17. When $\lambda \neq \emptyset$ and $k > 2$, $\mathfrak{t}^{(\lambda,l)} = \mathfrak{t}^{(\emptyset,l)} \circ \mathfrak{t}^{(\lambda,0)}[2l]$, and

$$d_{\mathfrak{t}^{(\lambda,l)}} = d_{\mathfrak{t}^{(\lambda,0)}[2l]} d_{\mathfrak{t}^{(\emptyset,l)}} = e_{k-2}^{(l)} e_{k-3}^{(l)} \cdots e_{2l-1}^{(l)} d_{\mathfrak{t}^{(\emptyset,l)}}.$$

If $l = 1$, $d_{\mathfrak{t}^{(\emptyset,l)}} = 1$ and the desired formula for $d_{\mathfrak{t}^{(\lambda,l)}}$ follows. If $l \geqslant 2$, $d_{\mathfrak{t}^{(\emptyset,l)}} = e_{2l-2}^{(l-1)} e_{2l-3}^{(l-1)}$, and the formula follows from applying Lemma 2.17.

(3) The element $e_{k-1}^{(l)} e_{k-2}^{(l)} e_{k-3}^{(l)} \cdots e_{2l-1}^{(l)}$ is a Brauer diagram with vertical strands connecting the top vertex $\boldsymbol{j}$ and the bottom vertex $\overline{\boldsymbol{j+2l}}$ for $1 \leqslant j \leqslant (k-2l)$. Therefore, assertion (3) follows from diagramatic computation. $\square$

Notation 2.19. For $(\lambda, l) \in \widehat{B}_k$, write $m_{(\lambda,l)} = d_{\mathfrak{t}^{(\lambda,l)}}^* c_{(\lambda,l)} d_{\mathfrak{t}^{(\lambda,l)}}$.

Remark 2.20. It follows from Lemma 2.18 parts (2) and (3) that

$$m_{(\lambda,l)} = e_{2l-1}^{(l)} c_{(\lambda,0)}[2l]. \tag{2.10}$$

3. Jucys–Murphy elements and the seminormal basis

3.1. **Nazarov's Jucys–Murphy elements.** Nazarov [16, Sect. 2] has defined a family of JM elements $(L_i)_{i\geqslant 0}$ for the tower $(\mathcal{B}_i)_{i\geqslant 0}$ by the relations

$$L_0 = L_1 = 0, \text{ and } L_{i+1} = s_i - e_i + s_i L_i s_i, \qquad \text{for } i \geqslant 1$$

Let $\mathscr{L}_k$ denote the subalgebra of $\mathcal{B}_k$ generated by $\{L_i \mid 1 \leqslant i \leqslant k\}$.

Proposition 3.1 (See [16, Sect. 2])**.** *Let $k \in \mathbb{Z}_{\geqslant 1}$. Then the following statements hold:*
(1) $L_0 + L_1 + \cdots + L_k$ is central in $\mathcal{B}_k$.
(2) L_k commutes with $\mathcal{B}_{k-1}$.
(3) The subalgebra $\mathscr{L}_k \subseteq \mathcal{B}_k$ is commutative.

Recall that if a is a cell in a Young diagram, the content $c(a)$ of a is the column index of a minus the row index of a. The Brauer algebra analogue is the following: for an edge $(\lambda, l) \to (\mu, m)$ in $\widehat{B}$, let

$$c((\lambda, l) \to (\mu, m)) = \begin{cases} c(a), & \text{if } \mu = \lambda \cup \{a\}, \\ 1 - \boldsymbol{z} - c(a), & \text{if } \mu = \lambda \setminus \{a\}. \end{cases} \tag{3.1}$$

For $(\lambda, l) \in \widehat{B}_k$ and $\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$, and for $1 \leqslant i \leqslant k$, define

$$c_{\mathfrak{t}}(i) = c(\mathfrak{t}^{(i-1)} \to \mathfrak{t}^{(i)}).$$

The following is the crucial property of the Jucys–Murphy elements and the Murphy bases of the cell modules.

Proposition 3.2 (See [9, Section 6.4])**.** *Let $(\lambda, l) \in \widehat{B}_k$ and $\mathfrak{t} = ((\lambda^{(0)}, l_0), \ldots, (\lambda^{(k)}, l_k)) \in \widehat{B}_k^{(\lambda,l)}$. If $i = 1, \ldots, k$, then there exist scalars $r_{\mathfrak{s}} \in R$, for $\mathfrak{s} \in \widehat{B}_k^{(\lambda,l)}$, such that*

$$m_{\mathfrak{t}} L_i = c_{\mathfrak{t}}(i) m_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} m_{\mathfrak{s}}.$$

Use of the reverse lexicographic order Proposition 3.2 repairs an error in [5].

The conclusion of Proposition 3.2 still holds for any specialization $B_k(S; \delta)$ of the Brauer algebras, where the variable $\boldsymbol{z}$ in (3.1) has to be replaced by δ. It follows from this that $\{L_1, \ldots, L_k\}$ is a family of Jucys–Murphy elements for $B_k(S; \delta)$ in the sense of Mathas [13, Definition 2.4]. One says that the Jucys–Murphy elements $\{L_1, \ldots, L_k\}$ *separate paths* (over S) if for $\mathfrak{s} \neq \mathfrak{t}$ in $\widehat{B}_k^{(\cdot)}$, there exists a j such that $c_{\mathfrak{t}}(j) \neq c_{\mathfrak{s}}(j)$; see [13, Definition 2.8]. Evidently, a sufficient condition for this is that S has characteristic zero and δ is not an integer. In particular, this holds for $B_k(\mathbb{F}; \boldsymbol{z})$. Mathas shows [13, Corollary 2.9] that if K is a field and if the the Jucys–Murphy elements $\{L_1, \ldots, L_k\}$ separate paths over K, then $B_k(K; \delta)$ is split semisimple.

Remark 3.3. This suggests a different proof of Wenzl's theorem: if K is a field of characteristic zero and $\delta \in K$ is not an integer, then $B_k(K; \delta)$ is split semisimple. The proof relies on [9, Section 6.4] and [13, Corollary 2.9], and avoids the determination of the weights of the Markov trace, as in [18]. We omit the details as the alternative proof is neither more efficient nor more elegant than the original proof from [18].

3.2. **Seminormal bases.** Let $\mathbb{F} = \mathbb{Q}(\boldsymbol{z})$ denote the field of fractions of $R = \mathbb{Z}[\boldsymbol{z}]$. For the remainder of Section 3, we denote $B_k(\mathbb{F}; \boldsymbol{z}) = \mathcal{B}_k(\boldsymbol{z}) \otimes_R \mathbb{F}$ by B_k. Later in the paper where we need to deal with several specializations of the Brauer algebras in the same context, we will write $B_k(\boldsymbol{z})$ for $B_k(\mathbb{F}, \boldsymbol{z})$. For $(\lambda, l) \in \widehat{B}_k$, denote the specialization of the cell module $\Delta_k^{(\lambda,l)} \otimes_R \mathbb{F}$ by $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$.

For each $i \geqslant 1$ let Let $\mathcal{C}(i)$ denote the set of Brauer contents associated to edges from level $i-1$ to level i in $\widehat{B}$,

$$\mathcal{C}(i) = \{c((\lambda, l) \to (\mu, m)) \mid (\lambda, l) \in \widehat{B}_{i-1}, (\mu, m) \in \widehat{B}_i, \text{ and } (\lambda, l) \to (\mu, m)\}.$$

Equivalently $\mathcal{C}(i)$ is the set of Brauer contents $c_{\mathfrak{t}}(i)$ for paths $\mathfrak{t}$ from $\emptyset$ to level k for any $k \geqslant i$.

Definition 3.4. Let $(\lambda, l) \in \widehat{B}_k$ and $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$. Define

$$F_{\mathfrak{t}} = \prod_{1 \leqslant i \leqslant k} \prod_{\substack{c \in \mathcal{C}(i) \\ c_{\mathfrak{t}}(i) \neq c}} \frac{L_i - c}{c_{\mathfrak{t}}(i) - c}$$

Definition 3.5. Let $f_{\mathfrak{t}} = m_{\mathfrak{t}} F_{\mathfrak{t}}$ and $F_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$.

Since the JM elements $L_1, \dots, L_k$ satisfy the separation condition of [13, Definition 2.8], as noted above, we obtain the following statements from [13, Sect. 3].

Proposition 3.6. *Let $k \geqslant 1$ and $(\lambda, l), (\mu, m) \in \widehat{B}_k$.*

(1) If $\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$, then there exist $r_{\mathfrak{s}} \in \mathbb{F}$, for $\mathfrak{s} \in \widehat{B}_k^{(\lambda,l)}$, such that

$$f_{\mathfrak{t}} = m_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} m_{\mathfrak{s}}. \tag{3.2}$$

(2) $\{f_{\mathfrak{t}} \mid \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}\}$ is an $\mathbb{F}$-basis for $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$.
(3) $\{F_{\mathfrak{s}\mathfrak{t}} \mid (\lambda, l) \in \widehat{B}_k$ and $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}\}$ is an $\mathbb{F}$-basis for B_k.
(4) $f_{\mathfrak{t}} L_i = c_{\mathfrak{t}}(i) f_{\mathfrak{t}}$, for all $\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$ and $i = 1, \dots, k$.
(5) $F_{\mathfrak{s}} F_{\mathfrak{t}} = \delta_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{s}}$ and $f_{\mathfrak{s}} F_{\mathfrak{t}} = \delta_{\mathfrak{s}\mathfrak{t}} f_{\mathfrak{s}}$, for all $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\cdot)}$.
(6) For $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$, $\langle f_{\mathfrak{s}}, f_{\mathfrak{t}} \rangle \neq 0$ if and only if $\mathfrak{s} = \mathfrak{t}$.
(7) For $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$ and $\mathfrak{u}, \mathfrak{v} \in \widehat{B}_k^{(\mu,m)}$, $F_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}\mathfrak{u}} \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle F_{\mathfrak{s}\mathfrak{v}}$, and $f_{\mathfrak{t}} F_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}\mathfrak{u}} \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle f_{\mathfrak{v}}$.
(8) $F_{\mathfrak{t}} = \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle^{-1} F_{\mathfrak{t}\mathfrak{t}}$ is a minimal indempotent, and $z_{(\lambda,l)} = \sum_{\mathfrak{s} \in \widehat{B}_k^{(\lambda,l)}} F_{\mathfrak{s}}$ is a minimal central idempotent. The ideal $z_{(\lambda,l)} B_k$ is a full matrix algebra with matrix units

$$\{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle^{-1} F_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}\}.$$

The bases given in (3) and (4) above are the *seminormal bases* for B_k and $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$ respectively. Observe that, as eigenvectors for the action of the JM family in B_k, the seminormal bases for $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$ and B_k are uniquely determined up to scaling factors in $\mathbb{F}$.

Because the transition matrix between the m–basis and the f–basis of the cell module $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$ is unitriangular with respect to the reverse lexicographic order $\succ$, the inverse matrix has the same property:

$$m_{\mathfrak{t}} = f_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r'_{\mathfrak{s}} f_{\mathfrak{s}}. \tag{3.3}$$

Recall the B_k–B_k bimodule isomorphism $\alpha_{(\lambda,l)} : B_k^{\trianglerighteq(\lambda,l)} / B_k^{\vartriangleright(\lambda,l)} \to (\Delta_{k,\mathbb{F}}^{(\lambda,l)})^* \otimes \Delta_{k,\mathbb{F}}^{(\lambda,l)}$. determined by $\alpha_{(\lambda,l)} : m_{\mathfrak{s}\mathfrak{t}} + B_k^{\vartriangleright(\lambda,l)} \mapsto m_{\mathfrak{s}}^* \otimes m_{\mathfrak{t}}$. One has $\alpha_{(\lambda,l)}^{-1}(f_{\mathfrak{s}}^* \otimes f_{\mathfrak{t}}) = F_{\mathfrak{s}\mathfrak{t}} + B_k^{\vartriangleright(\lambda,l)}$. It follows from Proposition 3.6 parts (3) and (7) that $\{F_{\mathfrak{s}\mathfrak{t}} \mid (\lambda, l) \in \widehat{B}_k$ and $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}\}$ is a cellular basis of B_k. Moreover, from [7, Lemma 2.3] or from [13, Lemma 3.3] we have that

$$B_k^{\trianglerighteq(\lambda,l)} = \mathrm{span}_{\mathbb{F}}\{F_{\mathfrak{s}\mathfrak{t}} \mid (\mu, m) \trianglerighteq (\lambda, l) \text{ and } \mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\mu,m)}\}$$

and similarly for $B_k^{\vartriangleright(\lambda,l)}$. From Proposition 3.6 parts (7) and (8), we have

$$z_{(\lambda,l)} B_k^{\vartriangleright(\lambda,l)} = 0. \tag{3.4}$$

Consequently if $a, b \in B_k^{\trianglerighteq(\lambda,l)}$ and $a \equiv b \mod B_k^{\triangleright(\lambda,l)}$, then $az_{(\lambda,l)} = bz_{(\lambda,l)}$.

Lemma 3.7. *Let* $(\lambda, l) \in \widehat{B}_k$.
(1) For fixed $\mathfrak{s} \in \widehat{B}_k^{(\lambda,l)}$, $I_\mathfrak{s} = \mathrm{span}_\mathbb{F}\{m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t} : \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}\}$ *is a right ideal of* B_k *and* $f_\mathfrak{t} \mapsto m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t}$ *is a* B_k*–module isomorphism from* $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$ *onto* $I_\mathfrak{s}$.
(2) For $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$,

$$m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t}m_{\mathfrak{t}\,\mathfrak{s}} = \langle f_\mathfrak{t}, f_\mathfrak{t}\rangle m_{\mathfrak{s}\,\mathfrak{s}}z_{(\lambda,l)}. \tag{3.5}$$

(3)

$$c_{(\lambda,l)}d_{\mathfrak{t}^{(\lambda,l)}}z_{(\lambda,l)} = c_{(\lambda,l)}d_{\mathfrak{t}^{(\lambda,l)}}F_{\mathfrak{t}^{(\lambda,l)}}. \tag{3.6}$$

(4)

$$F_{\mathfrak{t}^{(\lambda,l)}}m_{(\lambda,l)} = m_{(\lambda,l)}F_{\mathfrak{t}^{(\lambda,l)}} = m_{(\lambda,l)}z_{(\lambda,l)} = F_{\mathfrak{t}^{(\lambda,l)}\,\mathfrak{t}^{(\lambda,l)}}. \tag{3.7}$$

Proof. (1) (Compare [13, Corollary 3.10].) The map $x \mapsto \alpha_{(\lambda,l)}^{-1}(m_\mathfrak{s}^* \otimes x)$ is a B_k–module isomorphism from $\Delta_{k,\mathbb{F}}^{(\lambda,l)}$ into $B_k^{\trianglerighteq(\lambda,l)}/B_k^{\triangleright(\lambda,l)}$. The image of f_t under this map is $m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t}+B_k^{\triangleright(\lambda,l)}$. Therefore, for $b \in B$, if $f_\mathfrak{t}b = \sum r_\mathfrak{v} f_\mathfrak{v}$, then

$$m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t}b = \sum r_\mathfrak{v} m_{\mathfrak{s}\,\mathfrak{v}}F_\mathfrak{v} + y,$$

with $y \in B_k^{\triangleright(\lambda,l)}$. Now multiply by $z_{(\lambda,l)}$ and use (3.4) and Proposition 3.6 part (8) to get

$$m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t}b = \sum r_\mathfrak{v} m_{\mathfrak{s}\,\mathfrak{v}}F_\mathfrak{v},$$

which proves the claim.

(2) For any $b \in B_k$, and $\mathfrak{u}, \mathfrak{v} \in \widehat{B}_k^{(\lambda,l)}$, we have

$$m_{\mathfrak{s}\,\mathfrak{u}}bm_{\mathfrak{v}\,\mathfrak{s}} \equiv \langle m_\mathfrak{u} b, m_\mathfrak{v}\rangle m_{\mathfrak{s}\,\mathfrak{s}} \equiv \langle m_\mathfrak{u}, m_\mathfrak{v} b^*\rangle m_{\mathfrak{s}\,\mathfrak{s}} \mod \mathcal{B}_k^{\triangleright(\lambda,l)}.$$

In particular,

$$m_{\mathfrak{s}\,\mathfrak{t}}F_\mathfrak{t}m_{\mathfrak{t}\,\mathfrak{s}} = \langle f_\mathfrak{t}, f_\mathfrak{t}\rangle m_{\mathfrak{s}\,\mathfrak{s}} + y,$$

where $y \in B_k^{\triangleright(\lambda,l)}$. Multplying by $z_{(\lambda,l)}$ gives the result.

(3) We have $m_{\mathfrak{t}^{(\lambda,l)}} = f_{\mathfrak{t}^{(\lambda,l)}} = m_{\mathfrak{t}^{(\lambda,l)}}F_{\mathfrak{t}^{(\lambda,l)}}$, and hence

$$c_{(\lambda,l)}d_{\mathfrak{t}^{(\lambda,l)}} = c_{(\lambda,l)}d_{\mathfrak{t}^{(\lambda,l)}}F_{\mathfrak{t}^{(\lambda,l)}} + y,$$

where $y \in B_k^{\triangleright(\lambda,l)}$. Multplying by $z_{(\lambda,l)}$ gives the result.

(4) It follows from part (3) that $m_{(\lambda,l)}z_{(\lambda,l)} = m_{(\lambda,l)}F_{\mathfrak{t}^{(\lambda,l)}}$. Hence

$$m_{(\lambda,l)}F_{\mathfrak{t}^{(\lambda,l)}} = (m_{(\lambda,l)}F_{\mathfrak{t}^{(\lambda,l)}})^* = F_{\mathfrak{t}^{(\lambda,l)}}m_{(\lambda,l)}.$$

Now we have

$$m_{(\lambda,l)}z_{(\lambda,l)} = F_{\mathfrak{t}^{(\lambda,l)}}m_{(\lambda,l)}F_{\mathfrak{t}^{(\lambda,l)}} = F_{\mathfrak{t}^{(\lambda,l)}\,\mathfrak{t}^{(\lambda,l)}}.$$

□

We record the following special case of (3.5), which will be used repeatedly:

$$c_{(\lambda,l)}d_\mathfrak{t}F_\mathfrak{t}d_\mathfrak{t}^*c_{(\lambda,l)} = \langle f_\mathfrak{t}, f_\mathfrak{t}\rangle c_{(\lambda,l)}z_{(\lambda,l)}, \tag{3.8}$$

for $\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$. Indeed, according to Lemma 2.18 part (1), this is the special case with $\mathfrak{s} = \mathfrak{u}^{(\lambda,l)}$.

In the remainder of this subsection, we present some useful alternative expressions for the idempotents F_t. The following terminology is due to Vershik and Okounkov (find reference).

Definition 3.8. Fix $k \geqslant 1$. The *Gelfand Zeitlin* subalgebra of B_k is the maximal abelian subalgebra generated by the centers of $B_1, B_2, \ldots, B_k$. For $j \leqslant k$ and for $\mathfrak{t}$ a path on $\widehat{B}$ to level j, the *Gelfand Zeitlin* idempotent $F'(\mathfrak{t})$ is defined as $\prod_s z_{\mathfrak{t}(s)}$.

Proposition 3.9 ([9], Proposition 3.11)**.** *The Jucys–Murphy elements $L_1, \dots, L_k$ generate the Gelfand–Zeitlin subalgebra of B_k and the minimal idempotents F_t coincide with the Gelfand–Zeitlin idempotents F'_t.*

Corollary 3.10. *Let $1 \leqslant j \leqslant k$, let $\mathfrak{t} \in \widehat{B}^{(\cdot)}_k$ and $\mathfrak{s} \in \widehat{B}^{(\cdot)}_j$. Then $F_\mathfrak{t} F_\mathfrak{s} = \delta_{\mathfrak{t}_{\downarrow j},\mathfrak{s}} F_\mathfrak{t}$ and $f_\mathfrak{t} F_\mathfrak{s} = \delta_{\mathfrak{t}_{\downarrow j},\mathfrak{s}} f_\mathfrak{t}$.*

Recall that $\mathcal{C}(i)$ denotes the set of Brauer contents associated to edges from level $i-1$ to level i in $\widehat{B}$. For fixed $(\lambda, l) \in \widehat{B}_{i-1}$, let $\mathcal{C}(\lambda, i) \subseteq \mathcal{C}(i)$ denote the set of Brauer contents

$$\mathcal{C}(\lambda, i) = \{c((\lambda, l) \to (\mu, m)) \mid (\mu, m) \in \widehat{B}_i, \text{ and } (\lambda, l) \to (\mu, m)\}.$$

Lemma 3.11. *Fix an edge $(\lambda, l) \to (\mu, m)$ from level $i-1$ to level i in $\widehat{B}$. Write $c_0 = c((\lambda, l) \to (\mu, m))$. Then*

$$z(\lambda, l) \prod_{\substack{c \in \mathcal{C}(\lambda,i) \\ c_0 \neq c}} \frac{L_i - c}{c_0 - c} = z(\lambda, l) \prod_{\substack{c \in \mathcal{C}(i) \\ c_0 \neq c}} \frac{L_i - c}{c_0 - c} = z(\lambda, l) z(\mu, m). \tag{3.9}$$

Proof. Let $\mathfrak{s} \in \widehat{B}^{(\cdot)}_i$. Multiplying any of the three expression in (3.9) on the left by $F_\mathfrak{s}$ gives $F_\mathfrak{s}$ if $\mathfrak{s}^{(i-1)} = (\lambda, l)$ and $\mathfrak{s}^{(i)} = (\mu, m)$, and zero otherwise. Since $\sum_\mathfrak{s} F_\mathfrak{s} = 1$, the result follows. □

Corollary 3.12. *For any $\mathfrak{t} \in \widehat{B}^{(\cdot)}_k$,*

$$F_\mathfrak{t} = \prod_{1 \leqslant i \leqslant k} \prod_{\substack{c \in \mathcal{C}(\mathfrak{t}(i-1), i) \\ c_\mathfrak{t}(i) \neq c}} \frac{L_i - c}{c_\mathfrak{t}(i) - c} = F_{\mathfrak{t}'} \prod_{\substack{\mathfrak{s} \neq \mathfrak{t} \\ \mathfrak{s}' = \mathfrak{t}'}} \frac{L_k - c_\mathfrak{s}(k)}{c_\mathfrak{t}(k) - c_\mathfrak{s}(k)}.$$

Proof. This follows from Lemma 3.11 and Proposition 3.9. □

3.3. **Seminormal representations and restriction.** The seminormal representations are the representations of the Brauer algebras with respect to the seminormal bases of the cell modules. In this subsection, we consider the restriction of a seminormal representation of B_k to B_j and to the relative commutant of B_j in B_k, for $j < k$.

In the following, for $j < k$, write $B_k \cap B'_j$ for the set of $x \in B_k$ such that x commutes with all $y \in B_j$.

Lemma 3.13. *Let $1 \leqslant j < k$. Let $(\lambda, l) \in \widehat{B}_k$ and $\mathfrak{t} \in \widehat{B}^{(\lambda,l)}_k$. Let $(\mu, m) \in \widehat{B}_j$ and $\mathfrak{u}, \mathfrak{v} \in \widehat{B}^{(\mu,m)}_j$. Write $\mathfrak{t}_1 = \mathfrak{t}_{[0,j]}$ and $\mathfrak{t}_2 = \mathfrak{t}_{[j,k]}$, so $\mathfrak{t} = \mathfrak{t}_1 \circ \mathfrak{t}_2$. Then*

$$f_\mathfrak{t} F_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}_1 \mathfrak{u}} \langle f_{\mathfrak{t}_1}, f_{\mathfrak{t}_1} \rangle f_{\mathfrak{v} \circ \mathfrak{t}_2}.$$

Proof. We can embed the cell module $\Delta^{(\lambda,l)}_{k,\mathbb{F}}$ in B_k, identifying $f_\mathfrak{t}$ with $c_{(\lambda,l)} d_\mathfrak{t} F_\mathfrak{t}$, using Lemma 3.7 part (1) and taking $\mathfrak{s} = \mathfrak{u}^{(k,l)}$. Write $F_{\mathfrak{t}_2} = \prod_{j<n\leqslant k} z_{\mathfrak{t}(n)}$; then $F_\mathfrak{t} = F_{\mathfrak{t}_1} F_{\mathfrak{t}_2}$ and $F_{\mathfrak{t}_2} \in B_k \cap B'_j$. Thus,

$$f_\mathfrak{t} = c_{(\lambda,l)} d_\mathfrak{t} F_\mathfrak{t} = c_{(\lambda,l)} d_{\mathfrak{t}_2} d_{\mathfrak{t}_1} F_{\mathfrak{t}_1} F_{\mathfrak{t}_2}.$$

Using (2.6) and induction, we have an element $u_{\mathfrak{t}_2} \in B_k$ such that $c_{(\lambda,l)} d_{\mathfrak{t}_2} = u^*_{\mathfrak{t}_2} c_{\mathfrak{t}(j)}$. Inserting this, and using that $F_{\mathfrak{t}_2}$ commutes pointwise with B_j, we get

$$f_\mathfrak{t} = u^*_{\mathfrak{t}_2} F_{\mathfrak{t}_2} c_{\mathfrak{t}(j)} d_{\mathfrak{t}_1} F_{\mathfrak{t}_1}.$$

Applying Lemma 3.7 to B_j, we have

$$f_\mathfrak{t} F_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}_1 \mathfrak{u}} \langle f_{\mathfrak{t}_1}, f_{\mathfrak{t}_1} \rangle u^*_{\mathfrak{t}_2} F_{\mathfrak{t}_2} c_{\mathfrak{t}(j)} d_\mathfrak{v} F_\mathfrak{v}.$$

Now we can reverse the steps taken above to rewrite this as

$$f_\mathfrak{t} F_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}_1 \mathfrak{u}} \langle f_{\mathfrak{t}_1}, f_{\mathfrak{t}_1} \rangle c_{(\lambda,l)} d_{\mathfrak{v}\circ\mathfrak{t}_2} F_{\mathfrak{v}\circ\mathfrak{t}_2}.$$

$\square$

Proposition 3.14. *Let $1 \leqslant j < k$, $(\lambda, l) \in \widehat{B}_k$, and $\mathfrak{t} \in \widehat{B}_k^{(\lambda,l)}$. Write $\mathfrak{t}_1 = \mathfrak{t}_{[0,j]}$ and $\mathfrak{t}_2 = \mathfrak{t}_{[j,k]}$, so $\mathfrak{t} = \mathfrak{t}_1 \circ \mathfrak{t}_2$. Write (μ, m) for $\mathfrak{t}^{(j)}$.*

(1) If $x \in B_j$, then

$$f_{\mathfrak{t}} x = \sum_{\mathfrak{s}} r_{\mathfrak{s}} f_{\mathfrak{s} \circ \mathfrak{t}_2},$$

where $\mathfrak{s}$ ranges over $\widehat{B}_j^{(\mu,m)}$, and the coefficients $r_{\mathfrak{s}} \in \mathbb{F}$ depend only on x and $\mathfrak{t}_1$ and not on $\mathfrak{t}_2$ (or k).

(2) If $x \in B_k \cap B_j'$, then

$$f_{\mathfrak{t}} x = \sum_{\mathfrak{s}} r_{\mathfrak{s}} f_{\mathfrak{t}_1 \circ \mathfrak{s}},$$

where $\mathfrak{s}$ ranges over paths on $\widehat{B}$ from (μ, m) to (λ, l), and the coefficients $r_{\mathfrak{s}} \in \mathbb{F}$ depend only on x and $\mathfrak{t}_2$ and not on $\mathfrak{t}_1$.

Proof. Because $\{F_{\mathfrak{u}\,\mathfrak{v}}\}$, where $\mathfrak{u}, \mathfrak{v}$ are paths of length j with the same shape, is a basis of B_j, part (1) follows immediately from Lemma 3.13.

Now suppose that $x \in B_k \cap B_j'$. Write $f_{\mathfrak{t}} x = \sum_{\mathfrak{u}} r_{\mathfrak{u}} f_{\mathfrak{u}}$, where the sum is over $\mathfrak{u} \in \widehat{B}_k^{(\lambda,l)}$. Since x commutes with B_j,

$$f_{\mathfrak{t}} x = f_{\mathfrak{t}} F_{\mathfrak{t}_1} x = f_{\mathfrak{t}} x F_{\mathfrak{t}_1} = \sum_{\mathfrak{u}} r_{\mathfrak{u}} f_{\mathfrak{u}} F_{\mathfrak{t}_1},$$

which shows that $r_{\mathfrak{u}} = 0$ unless $\mathfrak{u}_{\downarrow j} = \mathfrak{t}_1$. Therefore, we rewrite the expansion of $f_{\mathfrak{t}} x$ as

$$f_{\mathfrak{t}} x = \sum_{\mathfrak{s}} r_{\mathfrak{s}} f_{\mathfrak{t}_1 \circ \mathfrak{s}},$$

where now the sum is over paths $\mathfrak{s}$ from (μ, m) to (λ, l). It remains to show that the coefficients $r_{\mathfrak{s}}$ do not depend on $\mathfrak{t}_1$. If $\mathfrak{v} \in \widehat{B}_j^{(\mu,m)}$, then

$$\begin{aligned} f_{\mathfrak{v} \circ \mathfrak{t}_2} x &= (1/\gamma) f_{\mathfrak{t}} F_{\mathfrak{t}_1\,\mathfrak{v}} x = (1/\gamma) f_{\mathfrak{t}} x F_{\mathfrak{t}_1\,\mathfrak{v}} \\ &= (1/\gamma) \sum_{\mathfrak{s}} r_{\mathfrak{s}} f_{\mathfrak{t}_1 \circ \mathfrak{s}} F_{\mathfrak{t}_1\,\mathfrak{v}} = \sum_{\mathfrak{s}} r_{\mathfrak{s}} f_{\mathfrak{v} \circ \mathfrak{s}}, \end{aligned}$$

where we have written $\gamma = \langle f_{\mathfrak{t}_1}, f_{\mathfrak{t}_1} \rangle$ and applied Lemma 3.13. $\square$

Corollary 3.15. *Assume that $(\lambda, l) \in \widehat{B}_k$ and $(\rho, r) \in \widehat{B}_{k-1}$, where $(\rho, r) \to (\lambda, l) \in \widehat{B}$. Let $N^{(\rho,r)} \subseteq \Delta_{k,\mathbb{F}}^{(\lambda,l)}$ denote the subspace with basis $\{f_{\mathfrak{s}} \in \Delta_{k,\mathbb{F}}^{(\lambda,l)} \mid \mathrm{Shape}(\mathfrak{s}_{\downarrow k-1}) = (\rho, r)\}$.*

(1) The linear map $N^{(\rho,r)} \to \Delta_{k-1,\mathbb{F}}^{(\rho,r)}$ given by $f_{\mathfrak{s}} \mapsto f_{\mathfrak{s}'}$ is an isomorphism of $B_{k-1}(\boldsymbol{z})$-modules.

(2) These maps induce an isomorphism of right $B_{k-1}(\boldsymbol{z})$–modules

$$\Delta_{k,\mathbb{F}}^{(\lambda,l)} \cong \bigoplus_{(\mu,m) \to (\lambda,l)} \Delta_{k-1,\mathbb{F}}^{(\mu,m)},$$

where the sum is over $(\mu, m) \in \widehat{B}_{k-1}$ such that $(\mu, m) \to (\lambda, l)$ in $\widehat{B}$.

Proof. Immediate from Lemma 3.14 part (1). $\square$

3.4. **Preliminary results on matrix coefficients of the generators.** Let $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$. For $1 \leqslant i \leqslant k$, define structure constants $e_i(\mathfrak{s}, \mathfrak{t}), s_i(\mathfrak{s}, \mathfrak{t}) \in \mathbb{F}$ by

$$f_{\mathfrak{t}} e_i = \sum_{\mathfrak{s}} e_i(\mathfrak{s}, \mathfrak{t}) f_{\mathfrak{s}} \qquad \text{and} \qquad f_{\mathfrak{t}} s_i = \sum_{\mathfrak{s}} s_i(\mathfrak{s}, \mathfrak{t}) f_{\mathfrak{s}}.$$

Definition 3.16. Let $(\lambda, l) \in \widehat{B}_k$ and $\mathfrak{s}, \mathfrak{t} \in \widehat{B}^{(\lambda,l)}$. Write $\mathfrak{s} \overset{i}{\sim} \mathfrak{t}$, and say that $\mathfrak{s}$ and $\mathfrak{t}$ are is i–equivalent, if $\mathfrak{s}^{(j)} = \mathfrak{t}^{(j)}$ whenever $j \neq i$.

The following statement is an immediate consequence of Proposition 3.14.

Lemma 3.17. *The coefficients $e_i(\mathfrak{s}, \mathfrak{t})$ and $s_i(\mathfrak{s}, \mathfrak{t})$ are zero unless $\mathfrak{s} \overset{i}{\sim} \mathfrak{t}$. Moreover the coeff- cients depend only on $\mathfrak{s}_{[i-1,i+1]}$ and $\mathfrak{t}_{[i-1,i+1]}$.*

Lemma 3.18. *For $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$ and for $1 \leqslant i \leqslant k$, we have $s_i(\mathfrak{s}, \mathfrak{t}) = 0 \Leftrightarrow s_i(\mathfrak{t}, \mathfrak{s}) = 0$ and $e_i(\mathfrak{s}, \mathfrak{t}) = 0 \Leftrightarrow e_i(\mathfrak{t}, \mathfrak{s}) = 0$.*

Proof. We give the proof for e_i. Since $e_i^* = e_i$, we have $\langle f_{\mathfrak{s}} e_i, f_{\mathfrak{t}} \rangle = \langle f_{\mathfrak{s}}, f_{\mathfrak{t}} e_i \rangle$. Using the orthogonality of the basis $\{f_u\}$, this gives $e_i(\mathfrak{t}, \mathfrak{s}) \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle = e_i(\mathfrak{s}, \mathfrak{t}) \langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle$. Since $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle$ and $\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle$ are non–zero, the assertion follows. □

In preparation for the next results, let us recall a notational convention. If $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{t} = ((\lambda^{(0)}, l_0), \ldots, (\lambda^{(k+1)}, l_{k+1})) \in \widehat{B}_{k+1}^{(\lambda,l)}$, we write $\mathfrak{t}^{(j)}$ for $(\lambda^{(j)}, l_j)$ and $\mathfrak{t}(j)$ for $\lambda^{(j)}$.

Lemma 3.19. *Assume that $(\lambda, l) \in \widehat{B}_{k+1}$ with $k \geqslant 1$, and $\mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$. Then $c_{\mathfrak{t}}(k) + c_{\mathfrak{t}}(k+1) = 1 - \boldsymbol{z}$ if and only if $\mathfrak{t}(k-1) = \mathfrak{t}(k+1)$.*

Proposition 3.20. *Assume that $(\lambda, l) \in \widehat{B}_{k+1}$ with $k \geqslant 1$, and $\mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$. Then the following statements are equivalent:*

(1) $\mathfrak{t}(k-1) = \mathfrak{t}(k+1)$.
(2) $F_{\mathfrak{t}} e_k \neq 0$.
(3) $f_{\mathfrak{t}} e_k \neq 0$.
(4) $e_k(\mathfrak{t}, \mathfrak{t}) \neq 0$.

Proof. The implications (4) $\implies$ (3) $\implies$ (2) are evident. Assume that $\mathfrak{t}(k-1) \neq \mathfrak{t}(k+1)$. Then $c_{\mathfrak{t}}(k) + c_{\mathfrak{t}}(k+1) \neq 1 + \boldsymbol{z}$, by Lemma 3.19. We have $F_{\mathfrak{t}}(L_k + L_{k+1}) = (c_{\mathfrak{t}}(k) + c_{\mathfrak{t}}(k+1)) F_{\mathfrak{t}}$, by Proposition 3.6, and $(L_k + L_{k+1}) e_k = (1 + \boldsymbol{z}) e_k$, by [16, Proposition 2.3]. Thus

$$(c_{\mathfrak{t}}(k) + c_{\mathfrak{t}}(k+1)) F_{\mathfrak{t}} e_k = F_{\mathfrak{t}}(L_k + L_{k+1}) e_k = (1 + \boldsymbol{z}) F_{\mathfrak{t}} e_k.$$

It follows that $F_{\mathfrak{t}} e_k = 0$. This give the implication (2) $\implies$ (1).

Now assume that $\mathfrak{t}(k-1) = \mathfrak{t}(k+1)$. Write $\mathfrak{t}' = \mathfrak{t}_{\downarrow k}$ and $\mathfrak{t}'' = \mathfrak{t}_{\downarrow k-1}$. We have $F_{\mathfrak{t}'} e_k = \sum \{F_{\mathfrak{s}} e_k : \mathfrak{s}' = \mathfrak{t}'\}$, using Proposition 3.9. By the previous paragraph, when $\mathfrak{s} \neq \mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{t}'$, it follows that $F_{\mathfrak{s}} e_k = 0$. Thus $F_{\mathfrak{t}'} e_k = F_{\mathfrak{t}} e_k$.

Note that $F_{\mathfrak{t}'} = z_{\mathfrak{t}(k)} F_{\mathfrak{t}''}$, by Proposition 3.9. Therefore, using Lemma 2.6, $\varepsilon_{k-1}(F_{\mathfrak{t}'}) = (\tau(F_{\mathfrak{t}'}) / \tau(F_{\mathfrak{t}''})) F_{\mathfrak{t}''}$. Multiplying the equation $F_{\mathfrak{t}} e_k = F_{\mathfrak{t}'} e_k$ by e_k on the left yields

$$e_k F_{\mathfrak{t}} e_k = e_k F_{\mathfrak{t}'} e_k = \boldsymbol{z} \varepsilon_{k-1}(F_{\mathfrak{t}'}) e_k = \boldsymbol{z} (\tau(F_{\mathfrak{t}'}) / \tau(F_{\mathfrak{t}''})) F_{\mathfrak{t}''} e_k.$$

But $F_{\mathfrak{t}''} e_k \neq 0$ by Lemma 2.4, so this this shows in particular that $F_t e_k \neq 0$. Multiplying the displayed equation on the left by F_t gives

$$e_k(\mathfrak{t}, \mathfrak{t}) F_{\mathfrak{t}} e_k = \boldsymbol{z} (\tau(F_{\mathfrak{t}'}) / \tau(F_{\mathfrak{t}''})) F_{\mathfrak{t}} e_k.$$

Since $F_{\mathfrak{t}} e_k \neq 0$, we have $e_k(\mathfrak{t}, \mathfrak{t}) = \boldsymbol{z} \tau(F_{\mathfrak{t}'}) / \tau(F_{\mathfrak{t}''}) \neq 0$. This shows (1) $\implies$ (4). □

Remark 3.21. A slightly different approach to this Proposition would use Lemma 2.5 instead of Lemma 3.19.

For the following corollary, recall that the weights of the Markov trace τ on $B_k(\boldsymbol{z})$ are given by the El Samra–King polynomials $P_\mu(\boldsymbol{z})$, see Theorem 2.8.

Corollary 3.22. *Let* $(\lambda, l) \in \widehat{B}_{k+1}$ *and* $\mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$, *with*

$$\mathfrak{t} = (\cdots, (\lambda, l-1), (\mu, m), (\lambda, l)).$$

Then

(1) $e_k F_\mathfrak{t} e_k = e_k(\mathfrak{t}, \mathfrak{t}) e_k F_{\mathfrak{t}''}$, *and*
(2) $e_k(\mathfrak{t}, \mathfrak{t}) = \boldsymbol{z}\tau(F_{\mathfrak{t}'})/\tau(F_{\mathfrak{t}''}) = P_\mu(\boldsymbol{z})/P_\lambda(\boldsymbol{z})$

Proof. The first statement and the first equality in (2) is contained in the proof of Proposition 3.20. The second equality in (2) results from applying Theorem 2.8. $\square$

Corollary 3.23. *Let* $(\lambda, l) \in \widehat{B}_{k+1}$ *and* $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$. *The matrix entry* $e_k(\mathfrak{s}, \mathfrak{t}) = 0$ *unless* $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$ *and* $\mathfrak{t}(k-1) = \mathfrak{t}(k+1)$.

Proof. This follows from Lemma 3.17 and Proposition 3.20. $\square$

Lemma 3.24. *Let* $(\lambda, l) \in \widehat{B}_{k+1}$ *and* $\mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in \widehat{B}_{k+1}^{(\lambda,l)}$, *where* $\mathfrak{s}(k-1) = \mathfrak{s}(k+1)$ *and* $\mathfrak{s}, \mathfrak{t}$, *and* $\mathfrak{u}$ *are* k*–equivalent paths.*

(1) $e_k(\mathfrak{s}, \mathfrak{t}) f_\mathfrak{s} e_k = e_k(\mathfrak{s}, \mathfrak{s}) f_\mathfrak{t} e_k$.
(2) $e_k(\mathfrak{u}, \mathfrak{s}) e_k(\mathfrak{s}, \mathfrak{t}) = e_k(\mathfrak{s}, \mathfrak{s}) e_k(\mathfrak{u}, \mathfrak{t})$.
(3) $e_k(\mathfrak{s}, \mathfrak{t}) \neq 0$

Proof. From Corollary 3.22, we have $e_k F_\mathfrak{s} e_k = e_k(\mathfrak{s}, \mathfrak{s}) e_k F_{\mathfrak{s}''}$. As in the proof of the previous lemma, apply this to $f_\mathfrak{t}$, but now observe that $f_\mathfrak{t} F_{\mathfrak{s}''} = f_\mathfrak{t}$ since $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$. This yields the equality $e_k(\mathfrak{s}, \mathfrak{t}) f_\mathfrak{s} e_k = e_k(\mathfrak{s}, \mathfrak{s}) f_\mathfrak{t} e_k$. Multiplying this on the right by $F_\mathfrak{u}$ gives $e_k(\mathfrak{s}, \mathfrak{t}) e_k(\mathfrak{u}, \mathfrak{s}) f_\mathfrak{u} = e_k(\mathfrak{s}, \mathfrak{s}) e_k(\mathfrak{u}, \mathfrak{t}) f_\mathfrak{u}$, hence the second assertion. In particular, we have $e_k(\mathfrak{s}, \mathfrak{t}) e_k(\mathfrak{t}, \mathfrak{s}) = e_k(\mathfrak{s}, \mathfrak{s}) e_k(\mathfrak{t}, \mathfrak{t})$, so $e_k(\mathfrak{s}, \mathfrak{t}) \neq 0$. $\square$

4. Seminormal representations

We continue to work over $\mathbb{F}$ and to write B_k for $B_k(\mathbb{F}, \boldsymbol{z})$. In this section, we obtain formulas for the matrix entries of the generators e_i and s_i with respect to the seminormal basis $\{f_\mathfrak{t}\}$ of the cell modules $\Delta_{k+1,\mathbb{F}}^{(\lambda,l)}$ of the Brauer algebras, as well as a branching rule for the inner products $\langle f_\mathfrak{t}, f_\mathfrak{t} \rangle$.

According to Lemma 3.17, for fixed $i \leqslant k$ the matrix coefficients of e_i and s_i with respect to the seminormal basis of $\Delta_{k+1,\mathbb{F}}^{(\lambda,l)}$ depend only on the initial portion of the paths indexing the seminormal basis, up to level $i+1$. Therefore, it suffices to find the matrix entries for e_k and s_k.

We begin by recalling Nazarov's formula [16, Corollary 3.10] for the diagonal matrix entries $e_k(\mathfrak{t}, \mathfrak{t})$ of the contractions e_k. Let $\mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$ with $\mathfrak{t}^{(k-1)} = (\lambda, l-1)$ and $\mathfrak{t}^{(k)} = (\mu, m)$. Write $b_0 = (z-1)/2 + c((\lambda, l-1) \to (\mu, m))$. Let B denote the set of elements

$$(z-1)/2 + c((\lambda, l-1) \to (\sigma, s)),$$

where $(\sigma, s) \in \widehat{B}_k$ and $(\lambda, l-1) \to (\sigma, s)$. Thus B is the set of eigenvaues of $L_k' = (z-1)/2 + L_k$, and b_0 the eigenvalue corresponding to the eigenvector $f_\mathfrak{t}$.

Proposition 4.1 ([16], Corollary 3.10). *Let $\mathfrak{t} \in \widehat{B}_{k+1}^{(\lambda,l)}$ with $\mathfrak{t}^{(k-1)} = (\lambda, l-1)$ and $\mathfrak{t}^{(k)} = (\mu, m)$. With the notation as in the previous paragraph,*

$$e_k(\mathfrak{t},\mathfrak{t}) = (2b_0+1) \prod_{b \in B \setminus \{b_0\}} \frac{b_0+b}{b_0-b} \tag{4.1}$$

Note that $e_k(\mathfrak{t},\mathfrak{t})$ is non–zero (by Lemma 3.20) and depends only on λ and μ. For example, if $\mathfrak{t}^{(k)} = (\mu, l-1)$ and $\mu = \lambda \cup \{a\}$, Nazarov's formula (4.1) translates to

$$e_k(\mathfrak{t},\mathfrak{t}) = (z+2c(a)) \frac{\prod_{\substack{\alpha \in A(\lambda)\\ \alpha \neq a}} (z-1+c(a)+c(\alpha))}{\prod_{\alpha \in R(\lambda)} (z-1+c(a)+c(\alpha))} \cdot \frac{\prod_{\alpha \in R(\lambda)} (c(a)-c(\alpha))}{\prod_{\substack{\alpha \in A(\lambda)\\ \alpha \neq a}} (c(a)-c(\alpha))}.$$

A similar formula holds in case $\mathfrak{t}^{(k)} = (\mu, l)$ and $\mu = \lambda \setminus \{a\}$.

Remark 4.2. Recall that we already have an alternative formula for $e_k(\mathfrak{t},\mathfrak{t})$ as a ratio of El Samra–King polynomials, $e_k(\mathfrak{t},\mathfrak{t}) = P_\mu(\boldsymbol{z})/P_\lambda(\boldsymbol{z})$, see Corollary 3.22 (2).

Lemma 4.3. *Let $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{t} = ((\lambda^{(0)}, l_0), \ldots, (\lambda^{(k+1)}, l_{k+1})) \in \widehat{B}_{k+1}^{(\lambda,l)}$. Assume that $\lambda^{(k+1)} \ominus \lambda^{(k-1)} = \{\alpha, \beta\}$.*

(1) If α and β are neither in the same row nor the same column, then there exists $\mathfrak{s} = \mathfrak{t}s_k \in \widehat{B}_{k+1}^{(\lambda,l)}$ such that $\{\mathfrak{u} \in \widehat{B}_{k+1}^{(\lambda,l)} \mid \mathfrak{u} \overset{k}{\sim} \mathfrak{t}\} = \{\mathfrak{s}, \mathfrak{t}\}$. Moreover, $c_{\mathfrak{s}}(k+1) = c_{\mathfrak{t}}(k)$ and $c_{\mathfrak{s}}(k) = c_{\mathfrak{t}}(k+1)$.

(2) If α and β are in the same row or the same column, then $\{\mathfrak{u} \in \widehat{B}_{k+1}^{(\lambda,l)} \mid \mathfrak{u} \overset{k}{\sim} \mathfrak{t}\} = \{\mathfrak{t}\}$.

Definition 4.4. Let λ be a partitions and $a = (i,j)$ be an addable or removable node of λ.

(1) Let $R(\lambda)^{<a} = \{(r, \lambda_r) \in R(\lambda) \mid r > i\}$.
(2) Let $A(\lambda)^{<a} = \{(r, \lambda_r + 1) \in A(\lambda) \mid r > i\}$.

The quantities in the following definition were used by James and Murphy [11] and Murphy [14] to compute the determinant of the Gram matrices for Specht modules for the symmetric groups. They will appear in the formulas for the matrix coefficients of the contractions e_i with respect to the seminormal basis in Theorem 4.11.

Definition 4.5. Let $\lambda \subset \mu$ be partitions with $\mu = \lambda \cup \{\alpha\}$. Define

$$\gamma_{\lambda \to \mu} = \frac{\prod_{\beta \in A(\lambda)^{<\alpha}} (c(\alpha)-c(\beta))}{\prod_{\beta \in R(\lambda)^{<\alpha}} (c(\alpha)-c(\beta))}. \tag{4.2}$$

Recall that h_α^μ denotes the hook length of a node α in a Young diagram μ, see Notation 2.7.

Lemma 4.6.

(1) Let $\lambda \subset \mu$ be partitions with $\mu = \lambda \cup \{\alpha\}$, where $\alpha = (j, \mu_j)$. Then

$$\gamma_{\lambda \to \mu} = \prod_{1 \leqslant k < \mu_j} \frac{h^\mu_{(j,k)}}{h^\mu_{(j,k)} - 1} \tag{4.3}$$

(2) Let σ be a partition with addable nodes α, β in different rows and columns, and with $\lambda = \sigma \cup \{\alpha\} \rhd \rho = \sigma \cup \{\beta\}$. Write $\mu = \sigma \cup \{\alpha, \beta\}$. Then

$$\frac{\gamma_{\lambda \to \mu}}{\gamma_{\sigma \to \rho}} = 1 - \frac{1}{(c(\alpha)-c(\beta))^2} \tag{4.4}$$

Proof. Part (1) is a straightforward exercise and part (2) follows since μ and ρ differ only in the node β. □

The next definition gives the Brauer algebra analogue of the branching coefficients $\gamma_{\lambda\to\mu}$.

Definition 4.7. Let $(\lambda, l) \in \widehat{B}_k$ and $(\mu, m) \in \widehat{B}_{k+1}$, with $(\lambda, l) \to (\mu, m)$ in $\widehat{B}$, Define

$$\gamma^{(k+1)}_{(\lambda,l)\to(\mu,m)} = \begin{cases} \gamma_{\lambda\to\mu}, & \text{if } l = m, \\ e_k(\mathfrak{t},\mathfrak{t})\gamma_{\mu\to\lambda}, & \text{if } l = m-1, \end{cases} \tag{4.5}$$

where, in the second case, $\mathfrak{t} \in \widehat{B}^{(\mu,m)}_{k+1}$ is any path which satisfies $\mathfrak{t}^{(k-1)} = (\mu, l)$ and $\mathfrak{t}^{(k)} = (\lambda, l)$.

As noted above, the structure constant $e_k(\mathfrak{t},\mathfrak{t})$ in (4.5) is non-zero and completely determined by λ and μ. Therefore $\gamma^{(k+1)}_{(\lambda,l)\to(\mu,m)}$depends only on λ and μ and not on m.

4.1. **Statement of the main results.** We are now ready to state the main results of this section. Theorem 4.8 and Theorem 4.11 together give complete formulae for the matrix entries of the generators s_i and e_i with respect to the seminormal basis of the cell modules $\Delta^{(\lambda,l)}_{k+1,\mathbb{F}}$. Some of these formulae were known. The diagonal entries are due to Nazarov (references). The expressions for structure constants (4.6) and (4.7) associated with the generator s_k are due to Rui and Si [17, Theorem 3.18(a)]. The formulae for the off-diagonal structure constants (4.10) and (4.11) associated with e_k and s_k appear to be new. Theorem 4.9 gives a recursion for the bilinear form $\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$. This result is of independent interest, but is also needed for the proof of Theorem 4.11. Finally, Theorem 4.10 gives a recursion for the determinant of the Gram matrix of the bilinear form (2.9) on the cell modules of the Brauer algebras.

Theorem 4.8 ([17], Theorem 3.18(a))**.** *Let $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{s},\mathfrak{t} \in \widehat{B}^{(\lambda,l)}_{k+1}$, where $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$.*

(1) If $\mathfrak{t}^{(k-1)} \neq (\lambda, l-1)$ and $\mathfrak{t}s_k$ does not exist, then

$$s_k(\mathfrak{s},\mathfrak{t}) = \frac{\delta_{\mathfrak{s}\mathfrak{t}}}{c_\mathfrak{t}(k+1) - c_\mathfrak{t}(k)}. \tag{4.6}$$

(2) If $\mathfrak{t}^{(k-1)} \neq (\lambda, l-1)$ and $\mathfrak{t}s_k$ exists, then

$$s_k(\mathfrak{s},\mathfrak{t}) = \begin{cases} \frac{1}{c_\mathfrak{t}(k+1)-c_\mathfrak{t}(k)}, & \text{if } \mathfrak{s} = \mathfrak{t}, \\ 1 - \frac{1}{(c_\mathfrak{t}(k+1)-c_\mathfrak{t}(k))^2}, & \text{if } \mathfrak{s} = \mathfrak{t}s_k \text{ and } \mathfrak{s} \succ \mathfrak{t}, \\ 1, & \text{if } \mathfrak{s} = \mathfrak{t}s_k \text{ and } \mathfrak{t} \succ \mathfrak{s}. \end{cases} \tag{4.7}$$

Theorem 4.9. *Let $(\lambda, l) \in \widehat{B}_k$ and $(\mu, m) \in \widehat{B}_{k+1}$, where $(\lambda, l) \to (\mu, m) \in \widehat{B}$. If $\mathfrak{s} \in \widehat{B}^{(\mu,m)}_{k+1}$ with $\mathfrak{s}^{(k)} = (\lambda, l)$, then*

$$\frac{\langle f_\mathfrak{s}, f_\mathfrak{s}\rangle}{\langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'}\rangle} = \gamma^{(k+1)}_{(\lambda,l)\to(\mu,m)}. \tag{4.8}$$

If $(\lambda, l) \in \widehat{B}_k$, let $G^{(\lambda,l)}_k$ denote the Gram matrix of the bilinear form (2.9) on $\Delta^{(\lambda,l)}_k$ and $\det\big(G^{(\lambda,l)}_k\big)$ denote the determinant of $G^{(\lambda,l)}_k$. The next statement, which is an immediate corollary of Theorem 4.9, gives a branching rule for the determinants $\det\big(G^{(\lambda,l)}_k\big)$. For the representations of the Brauer algebras which factor through the group algebra of the symmetric group, the recursion (4.9) coincides with the branching formula for the determinant of a Specht module given by James and Murphy [11, Sect. 2].

Theorem 4.10 (cf. [17, Theorem 4.11])**.** *Let $(\mu, m) \in \widehat{B}_{k+1}$. Then*

$$\det\big(G^{(\mu,m)}_{k+1}\big) = \prod_{(\lambda,l)\to(\mu,m)} \det\big(G^{(\lambda,l)}_k\big)\big(\gamma^{(k+1)}_{(\lambda,l)\to(\mu,m)}\big)^{\dim(\Delta^{(\lambda,l)}_k)}, \tag{4.9}$$

where the product is taken over $(\lambda, l) \in \widehat{B}_k$ such that $(\lambda, l) \to (\mu, m)$ in $\widehat{B}$.

Theorem 4.11. *Let $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{s}, \mathfrak{t} \in \widehat{B}^{(\lambda,l)}_{k+1}$, where $\mathfrak{s} \overset{k}{\sim} \mathfrak{t}$.*

(1) If $\mathfrak{s}^{(k-1)} \neq (\lambda, l-1)$, then $e_k(\mathfrak{s}, \mathfrak{t}) = 0$.

(2) If $\mathfrak{s}^{(k-1)} = (\lambda, l-1)$, then

$$e_k(\mathfrak{s}, \mathfrak{t}) = \begin{cases} \frac{\gamma_{\lambda\to\mu}}{\gamma_{\lambda\to\rho}} e_k(\mathfrak{t}, \mathfrak{t}), & \text{if } \mathfrak{s}^{(k)} = (\rho, l-1) \text{ and } \mathfrak{t}^{(k)} = (\mu, l-1), \\ \frac{\gamma_{\mu\to\lambda}}{\gamma_{\lambda\to\rho}}, & \text{if } \mathfrak{s}^{(k)} = (\rho, l-1) \text{ and } \mathfrak{t}^{(k)} = (\mu, l), \\ \frac{\gamma_{\lambda\to\mu}}{\gamma_{\rho\to\lambda}} e_k(\mathfrak{s}, \mathfrak{s}) e_k(\mathfrak{t}, \mathfrak{t}), & \text{if } \mathfrak{s}^{(k)} = (\rho, l) \text{ and } \mathfrak{t}^{(k)} = (\mu, l-1), \\ \frac{\gamma_{\mu\to\lambda}}{\gamma_{\rho\to\lambda}} e_k(\mathfrak{s}, \mathfrak{s}), & \text{if } \mathfrak{s}^{(k)} = (\rho, l) \text{ and } \mathfrak{t}^{(k)} = (\mu, l). \end{cases} \tag{4.10}$$

(3) If $\mathfrak{t}^{(k-1)} = (\lambda, l-1)$, then

$$s_k(\mathfrak{s}, \mathfrak{t}) = \frac{\delta_{\mathfrak{s}\mathfrak{t}} - e_k(\mathfrak{s}, \mathfrak{t})}{c_{\mathfrak{s}}(k+1) - c_{\mathfrak{t}}(k)}. \tag{4.11}$$

5. Proof of the branching formulae Theorem 4.9 and Theorem 4.10

In this subsection we verify the formula for the branching factors of the bilinear form $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle$ on cell modules given in Theorem 4.9. As a corollary, we obtain a branching formula for the Gram determinants of the cell modules.

Lemma 5.1. *Let $(\lambda, l) \in \widehat{B}_k$ and $(\mu, m) \in \widehat{B}_{k+1}$ with $(\lambda, l) \to \mu, m)$ in $\widehat{B}$. Consider paths $\mathfrak{s} \in \widehat{B}^{(\mu,m)}_{k+1}$ with $\mathfrak{s}^{(k)} = (\lambda, l)$. For any such path $\mathfrak{s}$, write $\mathfrak{s}' = \mathfrak{s}_{\downarrow k}$. Then the ratio $\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle / \langle f'_{\mathfrak{s}}, f'_{\mathfrak{s}} \rangle$ is independent of $\mathfrak{s}$, i.e. it depends only on the edge $(\lambda, l) \to (\mu, m)$.*

Proof. Write $d = d^{(k+1)}_{(\lambda,l)\to(\mu,m)}$ and $u = u^{(k+1)}_{(\lambda,l)\to(\mu,m)}$. Fix $\mathfrak{s} \in \widehat{B}^{(\mu,m)}_{k+1}$ with $\mathfrak{s}^{(k)} = (\lambda, l)$. We have

$$\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle c_{(\mu,m)} z_{(\mu,m)} = c_{(\mu,m)} d_{\mathfrak{s}} F_{\mathfrak{s}} d^*_{\mathfrak{s}} c_{(\mu,m)} = c_{(\mu,m)} d_{\mathfrak{s}} F_{\mathfrak{s}'} d^*_{\mathfrak{s}} c_{(\mu,m)} z_{(\mu,m)},$$

using (3.8) and the equality $F_{\mathfrak{s}} = F_{\mathfrak{s}'} z_{(\mu,m)}$. Continuing,

$$\begin{aligned} c_{(\mu,m)} d_{\mathfrak{s}} F_{\mathfrak{s}'} d^*_{\mathfrak{s}} c_{(\mu,m)} z_{(\mu,m)} &= u^* c_{(\lambda,l)} d_{\mathfrak{s}'} F_{\mathfrak{s}'} d^*_{\mathfrak{s}'} c_{(\lambda,l)} u z_{(\mu,m)} \\ &= \langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'} \rangle u^* c_{(\lambda,l)} z_{(\lambda,l)} u z_{(\mu,m)} = \langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'} \rangle c_{(\mu,m)} d z_{(\lambda,l)} u z_{(\mu,m)}. \end{aligned}$$

Thus

$$(\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle / \langle f'_{\mathfrak{s}}, f'_{\mathfrak{s}} \rangle) c_{(\mu,m)} z_{(\mu,m)} = c_{(\mu,m)} d z_{(\lambda,l)} u z_{(\mu,m)},$$

and the result follows since the right side depends only on the edge $(\lambda, l) \to \mu, m)$. □

Lemma 5.2. *Let $(\lambda, l) \in \widehat{B}_{k+1}$ and $\mathfrak{s} \in \widehat{B}^{(\lambda,l)}_{k+1}$. If $\mathfrak{t} = \mathfrak{s} s_k$ exists and $\mathfrak{t} \succ \mathfrak{s}$, then*

$$\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle = \frac{(c_{\mathfrak{s}}(k+1) - c_{\mathfrak{s}}(k))^2 - 1}{(c_{\mathfrak{s}}(k+1) - c_{\mathfrak{s}}(k))^2} \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle.$$

Proof. Apply the relation $\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle = \langle f_{\mathfrak{s}} s_k, f_{\mathfrak{s}} s_k \rangle$ and the formula (4.7). □

The following combinatorial lemma plays a crucial role.

Lemma 5.3. *Let $(\lambda, l) \in \widehat{B}_k$ and let $\alpha = (j, \lambda_j) \in R(\lambda)$. Denote $\mu = \lambda \setminus \{\alpha\}$. Let $a' = \sum_{r=1}^{j} \lambda_r$, $a = 2l + a'$ and $n = \sum_{r>j} \lambda_r$. Then the following statements hold:*

(1) If $0 \leqslant r \leqslant n$, then $\mathfrak{t}_r = (\cdots((\mathfrak{t}^{(\lambda,l)} s_a) s_{a+1}) \cdots) s_{a+r-1}$ exists in $\widehat{B}^{(\lambda,l)}_k$.

(2) The sequence $\{\mathfrak{t}_i \mid i = 0, \ldots, n\}$ satisfies $\mathfrak{t}_0 \succ \mathfrak{t}_1 \succ \cdots \succ \mathfrak{t}_n$.

(3) Write $\mathfrak{s} = \mathfrak{t}_n$. Then $\mathfrak{s}_{\downarrow k-1} = \mathfrak{t}^{(\mu,l)}$ and

$$f_{\mathfrak{t}^{(\lambda,l)}} s_a s_{a+1} \cdots s_{k-1} = f_{\mathfrak{s}} + \sum_{\mathfrak{u} \succ \mathfrak{s}} r_{\mathfrak{u}} f_{\mathfrak{u}}, \tag{5.1}$$

where the sum is over $\mathfrak{u} \in \widehat{B}_k^{(\lambda)}$ such that $\mathrm{Shape}(\mathfrak{u}_{\downarrow k-1}) \neq (\mu, l)$.

Proof. In case α is the lowest removable node of λ, all the statements are trivial. So we assume that α is not the lowest removable node, i.e., $\lambda_{j+1} > 0$

First we argue that we can reduce to the case $l = 0$, so the statement actually has to do with standard tableaux and the seminormal representations of the symmetric group. We have $\mathfrak{t} = \mathfrak{t}^{(\lambda,l)} = \mathfrak{t}^{(\emptyset,l)} \circ \mathfrak{t}^{(\lambda,0)}[2l]$. Consider the set of paths $P = \{\mathfrak{t}^{(\emptyset,l)} \circ \mathfrak{u}[2l]\}$, where $\mathfrak{u} \in \widehat{B}_{k-2l}^{(\lambda,0)}$. The (partial) action of $\mathfrak{S}\{a, \ldots, k\}$ on paths preserves P, since the hypotheses imply that $a \geqslant 2l + 2$. Moreover, it follows from (4.6) and (4.7) that the span of $\{f_{\mathfrak{v}} : \mathfrak{v} \in P\}$ is invariant under $\mathfrak{S}\{a, \ldots, k\}$, and the matrix coefficients for the action of $\mathfrak{S}\{a, \ldots, k\}$ are independent of l.

For the rest of the proof we suppose that $l = 0$, and we deal with the partial action of the symmetric group on standard tableaux, and the seminormal representation of $\mathfrak{S}_k$ corresponding to the partition λ. We have $\mathfrak{t}_0 = \mathfrak{t}^{\lambda}$, and for $1 \leqslant r \leqslant t$, the standard tableau $\mathfrak{t}_r = \mathfrak{t}^{\lambda}\, w_{a,a+r}$ is obtained by cyclically permuting the entries $a, \ldots, a+r$ in $\mathfrak{t}^{\lambda}$, so that $\mathfrak{t}_r(\alpha) = a + r$. Assertions (1) – (2) are evident, as is the statement that $(\mathfrak{t}_n)_{\downarrow k-1} = \mathfrak{t}^{\mu}$.

We turn to the proof of the final assertion of (3). With $\mathfrak{s} = \mathfrak{t}_n$, we have

$$f_{\mathfrak{t}^{\lambda}} s_a \cdots s_{k-1} = m_{\mathfrak{t}^{\lambda}} s_a \cdots s_{k-1} = m_{\mathfrak{s}} = f_{\mathfrak{s}} + \sum_{\mathfrak{u} \rhd \mathfrak{s}} r_{\mathfrak{u}} f_{\mathfrak{u}},$$

where $\rhd$ indicates dominance order on standard tableaux. On the other hand, it is evident from the restriction rule for the seminormal representations, i.e. the symmetric group analogue of Proposition 3.14, that those $\mathfrak{u}$ which appear with non–zero coefficients in the sum satisfy $\mathfrak{u}_{\downarrow a-1} = \mathfrak{t}^{\lambda}_{\downarrow a-1}$. It remains to verify that the $\mathfrak{u}$ appearing in the sum satisfy $\mathrm{node}_{\mathfrak{u}}(k) \neq \alpha$. We prove this by induction on n. If $n = 1$, then $f_{\mathfrak{t}^{\lambda}} s_{k-1} = f_{\mathfrak{s}} + \kappa f_{\mathfrak{t}^{\lambda}}$, by (4.7), so the assertion is valid. Suppose that $n > 1$. Let β denote $\mathrm{node}_{\mathfrak{t}^{\lambda}}(k) = (\ell, \lambda_{\ell})$, where $\ell = \mathrm{length}(\lambda)$. By the appropriate induction hypothesis and the restriction rule,

$$f_{\mathfrak{t}^{\lambda}} s_a \cdots s_{k-2} = f_{\mathfrak{t}_{n-1}} + \sum_{\mathfrak{v}} \kappa_{\mathfrak{v}} f_{\mathfrak{v}},$$

where the $\mathfrak{v}$ appearing in the sum satisfy $\mathrm{node}_{\mathfrak{v}}(k-1) \neq \alpha$, i.e., $\mathrm{row}_{\mathfrak{v}}(k-1) > j$, and $\mathrm{node}_{\mathfrak{v}}(k) = \beta$. Now multiply both sides by s_{k-1}. We have $f_{\mathfrak{t}_{n-1}} s_{k-1} = f_{\mathfrak{s}} + \gamma f_{\mathfrak{t}_{n-1}}$ by (4.7), and $\mathrm{node}_{\mathfrak{t}_{n-1}}(k) = \beta$. For the remaining terms on the right side, both $k-1$ and k have row index $> j$ in $\mathfrak{v}$, and therefore $f_{\mathfrak{v}} s_{k-1}$ is a linear combination of basis elements $f_{\mathfrak{u}}$ with the same property. □

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a partition, let $\lambda! = \prod_{r=1}^{t} (\lambda_r!)$.

Lemma 5.4. *If $(\lambda, l) \in \widehat{B}_k$, then $\langle f_{\mathfrak{t}^{(\lambda,l)}}, f_{\mathfrak{t}^{(\lambda,l)}} \rangle = \lambda! z^l$.*

Proof. From (3.5) and (3.7),

$$m_{(\lambda,l)}^2 z_{(\lambda,l)} = m_{(\lambda,l)} F_{\mathfrak{t}^{(\lambda,l)}} m_{(\lambda,l)} = \langle f_{\mathfrak{t}^{(\lambda,l)}}, f_{\mathfrak{t}^{(\lambda,l)}} \rangle m_{(\lambda,l)} z_{(\lambda,l)}. \tag{5.2}$$

On the other hand, $m_{(\lambda,l)} = e_1 e_3 \cdots e_{2l-1} c_{(\lambda,0)}[2l]$, from which it follows that $m_{(\lambda,l)}^2 = \boldsymbol{z}^l \lambda! m_{(\lambda,l)}$. Multiply this by $z_{(\lambda,l)}$ and compare with (5.2). □

Lemma 5.5. *Let $(\mu, l+1) \in \widehat{B}_{k+1}$ and let $\nu = \mu \cup \{(\ell + 1, 1)\}$, where $\ell = \mathrm{length}(\mu)$. Define $\mathfrak{u} \in \widehat{B}_{k+1}^{(\mu,l+1)}$ by the condition $\mathfrak{u}_{\downarrow k} = \mathfrak{t}^{(\nu,l)}$.*

(1) For $x \in B_{k-2l-1}$,

$$d_{\mathfrak{u}}^* x e_k^{(l+1)} d_{\mathfrak{u}} = e_{2l-1}^{(l)} x[2l] e_k. \tag{5.3}$$

(2) In particular

$$m_{\mathfrak{u}\,\mathfrak{u}} = d_{\mathfrak{u}}^* c_{(\mu,l+1)} d_{\mathfrak{u}} = m_{(\mu,l)} e_k. \tag{5.4}$$

Proof. We have $d_{\mathfrak{u}} = e_{k-1}^{(l)} d_{\mathfrak{t}^{(\nu,l)}}$. If $l = 0$, then $d_{\mathfrak{u}} = 1$, and assertion (1) is evident. If $l \geqslant 1$ then

$$\begin{aligned} d_{\mathfrak{u}}^* x e_k^{(l+1)} d_{\mathfrak{u}} &= e_{2l-1}^{(l)} \cdots e_{k-1}^{(l)} e_k^{(l+1)} x e_{k-1}^{(l)} \cdots e_{2l-1}^{(l)} \\ &= e_{2l-1}^{(l)} \cdots e_{k-1}^{(l)} e_k^{(l+1)} e_{k-1}^{(l)} \cdots e_{2l-1}^{(l)} x[2l] = e_{2l-1}^{(l)} x[2l] e_k, \end{aligned}$$

where we have used Lemma 2.18 part (3) and contraction identities. This completes the proof of statement (1) and statement (2) follows. □

Proof of Theorem 4.9. Recall the notation in the statement of the theorem: We have $\mathfrak{s} \in \widehat{B}_{k+1}^{(\mu,m)}$ with $\mathfrak{s}^{(k)} = (\lambda, l)$. We write $\mathfrak{s}' = \mathfrak{s}_{\downarrow k}$ and $\mathfrak{s}'' = \mathfrak{s}_{\downarrow k-1}$. We compute the quotient $\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle / \langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'} \rangle$ in two cases below.

CASE 1. Suppose $l = m$ and $\mu = \lambda \cup \{\alpha\}$, with $\alpha = (j, \mu_j)$. If $\mathfrak{s}$ is maximal with respect to $\succcurlyeq$, then $\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle / \langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'} \rangle = \mu_j$ by Lemma 5.4, and there is nothing further to show. We therefore proceed by a double induction on k and $\succcurlyeq$. If α is in the last row of μ, then by Lemma 5.1, we can assume that $\mathfrak{s} = \mathfrak{t}^{(\mu,m)}$, which is the case already covered. We may therefore assume that $\mu_{j+1} > 0$, and again by Lemma 5.1, we may assume without loss of generality that $\mathfrak{s}' = \mathfrak{t}^{(\lambda,m)}$. Since $\mu_{j+1} > 0$, in particular, $\lambda \neq \emptyset$, and thus $\mathfrak{s}^{(k-1)} = (\sigma, m)$, where $\sigma = \lambda \setminus \{\beta\}$ and $\beta = (r, \mu_r)$ is the removable node in the last row of λ. Let $\rho = \sigma \cup \{\alpha\}$ and $\mathfrak{t} = \mathfrak{s}'' \circ ((\rho, m), (\mu, m)) = \mathfrak{s} s_k$. Note that $\mathfrak{t} \succ \mathfrak{s}$. Let $\mathfrak{t}' = \mathfrak{t}_{\downarrow k}$. By the induction assumption on $\succcurlyeq$, we have

$$\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle = \mu_r \langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'} \rangle. \tag{5.5}$$

By Lemma 5.2 we have

$$\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle = (1 - \frac{1}{(c(\alpha) - c(\beta))^2}) \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle \tag{5.6}$$

By the induction assumption on k, we have

$$\frac{\langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'} \rangle}{\langle f_{\mathfrak{s}''}, f_{\mathfrak{s}''} \rangle} = \mu_r \quad \text{and} \quad \frac{\langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'} \rangle}{\langle f_{\mathfrak{s}''}, f_{\mathfrak{s}''} \rangle} = \gamma_{\sigma \to \rho}. \tag{5.7}$$

Combining equations (5.5) to (5.7) and using Lemma 4.6 part (2) yields

$$\frac{\langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle}{\langle f_{\mathfrak{s}'}, f_{\mathfrak{s}'} \rangle} = (1 - \frac{1}{(c(\alpha) - c(\beta))^2}) \gamma_{\sigma \to \rho} = \gamma_{\lambda \to \mu}, \tag{5.8}$$

as required.

CASE 2. Assume that $m = l + 1$ and let $\lambda = \mu \cup \{(j, \lambda_j)\}$. By Lemma 5.1, there is no loss of generality in assuming that $\mathfrak{s}' = \mathfrak{t}^{(\lambda,l)} \in \widehat{B}_k^{(\lambda,l)}$. Let $\nu = \mu \cup \{(\ell + 1, 1)\}$, where $\ell = \mathrm{length}(\mu)$. Define $\mathfrak{u} \in \widehat{B}_{k+1}^{(\mu,m)}$ by the requirement $\mathfrak{u}_{\downarrow k} = \mathfrak{t}^{(\nu,l)}$. We have $d_{\mathfrak{u}} = e_{k-1}^{(l)} d_{\mathfrak{t}^{(\nu,l)}}$, and $d_{\mathfrak{s}} = d_{(\lambda,l)\to(\mu,l+1)}^{(k+1)} d_{\mathfrak{t}^{(\lambda,l)}}$. Note that $d_{\mathfrak{t}^{(\nu,l)}} = d_{\mathfrak{t}^{(\lambda,l)}}$, since these quantities depend only on k and l. Thus,

$$\begin{aligned} m_{\mathfrak{u}\,\mathfrak{s}} &= d_{\mathfrak{t}^{(\lambda,l)}}^* e_{k-1}^{(l)} c_{(\mu,l+1)} d_{(\lambda,l)\to(\mu,l+1)}^{(k+1)} d_{\mathfrak{t}^{(\lambda,l)}} = d_{\mathfrak{t}^{(\lambda,l)}}^* e_{k-1}^{(l)} (u_{(\lambda,l)\to(\mu,l+1)}^{(k+1)})^* c_{(\lambda,l)} d_{\mathfrak{t}^{(\lambda,l)}} \\ &= d_{\mathfrak{t}^{(\lambda,l)}}^* e_{k-1}^{(l)} e_k^{(l+1)} (d_{\mu\to\lambda}^{(k-2l)})^* c_{(\lambda,l)} d_{\mathfrak{t}^{(\lambda,l)}}. \end{aligned}$$

We claim that

$$m_{\mathfrak{u}\,\mathfrak{s}} = e_k w_{k,a} m_{(\lambda,l)}, \tag{5.9}$$

where $a=\sum_{r=1}^{j}\lambda_r+2l$. The case $l=0$ is easy to check. If $l\geqslant 1$, then $d_{\mathfrak{t}(\lambda,l)}=e_{k-2}^{(l)}\cdots e_{2l-1}^{(l)}$, and

$$\begin{aligned} m_{\mathfrak{u}\,\mathfrak{s}} &= e_{2l-1}^{(l)}\cdots e_{k-1}^{(l)}e_k^{(l+1)}(d_{\mu\to\lambda}^{(k-2l)})^* c_{(\lambda,0)}e_{k-1}^{(l)}\cdots e_{2l-1}^{(l)} \\ &= e_{2l-1}^{(l)}\cdots e_{k-1}^{(l)}e_k^{(l+1)}e_{k-1}^{(l)}\cdots e_{2l-1}^{(l)}(d_{\mu\to\lambda}^{(k-2l)})^*[2l]c_{(\lambda,0)}[2l] \\ &= e_{2l-1}^{(l)}e_k w_{k,a}c_{(\lambda,0)}[2l]=e_k w_{k,a}m_{(\lambda,l)}. \end{aligned}$$

We have

$$m_{\mathfrak{u}\,\mathfrak{s}}F_{\mathfrak{s}}m_{\mathfrak{s}\,\mathfrak{u}}=\langle f_{\mathfrak{s}},f_{\mathfrak{s}}\rangle m_{\mathfrak{u}\,\mathfrak{u}}z_{(\mu,m)}, \tag{5.10}$$

by (3.5). On the other hand we compute, using $F_{\mathfrak{s}}=F_{\mathfrak{s}'}z_{(\mu,m)}$, Lemma 3.7 and equation (5.9),

$$\begin{aligned} m_{\mathfrak{u}\,\mathfrak{s}}F_{\mathfrak{s}}m_{\mathfrak{s}\,\mathfrak{u}} &= e_k w_{k,a}m_{(\lambda,l)}F_{\mathfrak{s}}m_{(\lambda,l)}w_{a,k}e_k=e_k w_{k,a}m_{(\lambda,l)}F_{\mathfrak{s}'}m_{(\lambda,l)}w_{a,k}e_k z_{(\mu,m)} \\ &= \langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle e_k w_{k,a}m_{(\lambda,l)}z_{(\lambda,l)}w_{a,k}e_k z_{(\mu,m)}=\langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle e_k w_{k,a}F_{\mathfrak{s}'\,\mathfrak{s}'}w_{a,k}e_k z_{(\mu,m)}. \end{aligned}$$

Let $\mathfrak{t}'=\mathfrak{s}'s_a s_{a+1}\cdots s_{k-1}$. By Lemma 5.3, $\mathfrak{t}'\in\widehat{B}_k^{(\lambda,l)}$ with $\mathfrak{t}''=\mathfrak{t}'_{\downarrow k-1}=\mathfrak{t}^{(\mu,l)}$, and

$$f_{\mathfrak{s}'}w_{a,k}=f_{\mathfrak{t}'}+\sum_{\mathfrak{a}}r_{\mathfrak{a}}f_{\mathfrak{a}},$$

where the sum is taken over $\mathfrak{a}\in\widehat{B}_k^{(\lambda,l)}$ such that $\mathrm{Shape}(\mathfrak{a}_{\downarrow k-1})\neq(\mu,l)$. It follows that

$$e_k w_{k,a}F_{\mathfrak{s}'\,\mathfrak{s}'}w_{a,k}e_k=e_kF_{\mathfrak{t}'\,\mathfrak{t}'}e_k=\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle e_kF_{\mathfrak{t}'}e_k.$$

Let $\mathfrak{t}=\mathfrak{t}'\circ((\mu,m))$. Using $F_{\mathfrak{t}'}z_{(\mu,m)}=F_{\mathfrak{t}}$, we have

$$\begin{aligned} m_{\mathfrak{u}\,\mathfrak{s}}F_{\mathfrak{s}}m_{\mathfrak{s}\,\mathfrak{u}} &= \langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle e_kF_{\mathfrak{t}}e_kz_{(\mu,m)} \\ &= \langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle e_k(\mathfrak{t},\mathfrak{t})F_{\mathfrak{t}''}e_kz_{(\mu,m)} \\ &= \langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle\frac{\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle}{\langle f_{\mathfrak{t}''},f_{\mathfrak{t}''}\rangle}e_k(\mathfrak{t},\mathfrak{t})F_{\mathfrak{t}''\,\mathfrak{t}''}e_kz_{(\mu,m)} \end{aligned}$$

Note that

$$m_{\mathfrak{u}\,\mathfrak{u}}z_{(\mu,m)}=m_{(\mu,l)}e_kz_{(\mu,m)}=m_{(\mu,l)}e_kz_{(\mu,l)}z_{(\mu,m)}=F_{\mathfrak{t}''\,\mathfrak{t}''}e_kz_{(\mu,m)},$$

where the first equality is from Lemma 5.5, the second equality from Proposition 3.20, and the last equality from (3.7), taking into account that $\mathfrak{t}''=\mathfrak{t}^{(\mu,l)}$. By Case 1 above, $\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle/\langle f_{\mathfrak{t}''},f_{\mathfrak{t}''}\rangle=\gamma_{\mu\to\lambda}$, and hence we have

$$m_{\mathfrak{u}\,\mathfrak{s}}F_{\mathfrak{s}}m_{\mathfrak{s}\,\mathfrak{u}}=\langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle\gamma_{\mu\to\lambda}e_k(\mathfrak{t},\mathfrak{t})m_{\mathfrak{u}\,\mathfrak{u}}z_{(\mu,m)}. \tag{5.11}$$

Comparing (5.10) and (5.11) gives the desired result,

$$\langle f_{\mathfrak{s}},f_{\mathfrak{s}}\rangle/\langle f_{\mathfrak{s}'},f_{\mathfrak{s}'}\rangle=\gamma_{\mu\to\lambda}e_k(\mathfrak{t},\mathfrak{t}).$$

□

The proof of Theorem 4.11 requires the following observation.

Corollary 5.6. *Let $(\lambda,l)\in\widehat{B}_k$ and $(\mu,l)\in\widehat{B}_{k-1}$, with $\lambda=\mu\cup\{(j,\lambda_j)\}$. Let $\mathfrak{t}$ denote $\mathfrak{t}^{(\lambda,l)}$ and $a=2l+\sum_{r=1}^{j}\lambda_r$. Let $\mathfrak{s}=\mathfrak{t}w_{a,k}$. Then*

$$f_{\mathfrak{s}}w_{k,a}=\frac{\gamma_{\mu\to\lambda}}{\lambda_j}f_{\mathfrak{t}}+\sum_{\mathfrak{u}\prec\mathfrak{t}}r_{\mathfrak{u}}f_{\mathfrak{u}}.$$

Proof. By Lemma 5.3, $\mathfrak{s}=\mathfrak{t}w_{a,k}$ exists in $\widehat{B}_k^{(\lambda,l)}$, and $\langle f_{\mathfrak{s}}w_{k,a},f_{\mathfrak{t}}\rangle=\langle f_{\mathfrak{s}},f_{\mathfrak{t}}w_{a,k}\rangle=\langle f_{\mathfrak{s}},f_{\mathfrak{s}}\rangle$. Therefore, the coefficient of f_t in the expansion of $f_{\mathfrak{s}}w_{k,a}$ in the seminormal basis is

$\langle f_\mathfrak{s}, f_\mathfrak{s}\rangle/\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$. It follows from Lemma 5.3 that $\mathfrak{s}' = \mathfrak{t}^{(\mu,m)}$. Therefore, using Theorem 4.9 and Lemma 5.4, we have

$$\langle f_\mathfrak{s}, f_\mathfrak{s}\rangle/\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle = \frac{\gamma_{\mu\to\lambda}\boldsymbol{z}^m\mu!}{\boldsymbol{z}^m\lambda!} = \frac{\gamma_{\mu\to\lambda}}{\lambda_j}.$$

□

Finally, we indicate how Theorem 4.10 follows from Theorem 4.9.

Proof of Theorem 4.10. The Gram determinant is the determinant of the matrix $[\langle m_\mathfrak{s}, m_\mathfrak{t}\rangle]_{\mathfrak{s},\mathfrak{t}}$. Since the transition matrix from the basis $\{m_\mathfrak{t}\}$ to the basis $\{f_\mathfrak{t}\}$ is unitriangular with respect the partial order $\succ$, and the basis elements $f_\mathfrak{t}$ are mutually orthogonal with respect to the bilinear form $\langle\cdot,\cdot\rangle$, this is the same as the product $\prod_\mathfrak{t}\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$. Now the recursion formula (4.9) follows from (4.8). □

6. Proof of the formulae for the seminormal representations Theorem 4.11

Proposition 6.1. *Let $(\lambda, l) \in \widehat{B}_{k+1}$, with $\lambda \neq \emptyset$ and $l \geqslant 1$. Let ℓ denote the length of λ, and let $\sigma = \lambda \setminus \{(\ell, \lambda_\ell)\}$ and $\nu = \lambda \cup \{(\ell+1, 1)\}$. Define $\mathfrak{s} \in \widehat{B}_{k+1}^{(\lambda,l)}$ by the conditions $\mathfrak{s}^{(k)} = (\sigma, l)$ and $\mathfrak{s}_{\downarrow k-1} = \mathfrak{t}^{(\lambda,l-1)}$. Define $\mathfrak{u} \in \widehat{B}_{k+1}^{(\lambda,l)}$ by the conditions $\mathfrak{u}^{(k)} = (\nu, l-1)$ and $\mathfrak{u}_{\downarrow k-1} = \mathfrak{t}^{(\lambda,l-1)}$. Then the following statements hold:*

(1) $f_\mathfrak{s} e_k = \gamma_{\sigma\to\lambda} m_\mathfrak{u}$ and $f_\mathfrak{u} e_k = e_k(\mathfrak{u},\mathfrak{u}) m_\mathfrak{u}$.

(2) If $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$ and $\mathfrak{t}^{(k)} = (\mu, l-1)$, where $(\mu, l-1) \in \widehat{B}_k$ and $(\mu, l-1) \to (\lambda, l)$, then

$$e_k(\mathfrak{s},\mathfrak{t}) = e_k(\mathfrak{s},\mathfrak{s}) e_k(\mathfrak{t},\mathfrak{t})\frac{\gamma_{\lambda\to\mu}}{\gamma_{\sigma\to\lambda}} \qquad \text{and} \qquad e_k(\mathfrak{t},\mathfrak{s}) = \frac{\gamma_{\sigma\to\lambda}}{\gamma_{\lambda\to\mu}}.$$

(3) If $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$ and $\mathfrak{t}^{(k)} = (\mu, l)$, where $(\mu, l) \in \widehat{B}_k$ and $(\mu, l) \to (\lambda, l)$, then

$$e_k(\mathfrak{s},\mathfrak{t}) = \frac{\gamma_{\mu\to\lambda}}{\gamma_{\sigma\to\lambda}} e_k(\mathfrak{s},\mathfrak{s}) \qquad \text{and} \qquad e_k(\mathfrak{t},\mathfrak{s}) = \frac{\gamma_{\sigma\to\lambda}}{\gamma_{\mu\to\lambda}} e_k(\mathfrak{t},\mathfrak{t}).$$

Proof. Let P denote the set of paths $\mathfrak{v} \in \widehat{B}_{k+1}^{(\lambda,l)}$ such that $\mathfrak{v}_{\downarrow k-1} = \mathfrak{t}^{(\lambda,l-1)}$. Remark that $\mathfrak{s}$ is the maximum element of P and $\mathfrak{u}$ is the minimal element of P, with respect to reverse lexicographic order.

(1) Note that

$$c_{(\lambda,0)} u_{\sigma\to\lambda}^{(k+1-2l)} = \lambda_\ell c_{(\lambda,0)} = \gamma_{\sigma\to\lambda} c_{(\lambda,0)}, \tag{6.1}$$

because $u_{\sigma\to\lambda}^{(k+1-2l)}$ is the sum of λ_ℓ elements of the row group $\mathfrak{S}_\lambda$. Using the definitions,

$$d_\mathfrak{s} = e_{k-1}^{(l)} u_{\sigma\to\lambda}^{(k+1-2l)} e_{k-2}^{(l-1)} d_{\mathfrak{t}^{(\lambda,l-1)}}.$$

If follows from the commutation and contraction relations and (6.1) that

$$c_{(\lambda,l)} d_\mathfrak{s} e_k = c_{(\lambda,0)} u_{\sigma\to\lambda}^{(k+1-2l)} e_k^{(l)} d_{\mathfrak{t}^{(\lambda,l-1)}} = \gamma_{\sigma\to\lambda} e_k c_{(\lambda,l-1)} d_{\mathfrak{t}^{(\lambda,l-1)}}. \tag{6.2}$$

Likewise,

$$d_\mathfrak{u} = e_{k-1}^{(l-1)} e_{k-2}^{(l-1)} d_{\mathfrak{t}^{(\lambda,l-1)}},$$

so

$$c_{(\lambda,l)} d_\mathfrak{u} = e_k c_{(\lambda,l-1)} d_{\mathfrak{t}^{(\lambda,l-1)}}. \tag{6.3}$$

Comparing (6.2) and (6.3), we get $m_\mathfrak{s} e_k = \gamma_{\sigma\to\lambda} m_\mathfrak{u}$.

Let $m_{\mathfrak{s}} = f_{\mathfrak{s}} + \sum_{\mathfrak{v}\succ\mathfrak{s}} r_{\mathfrak{v}} f_{\mathfrak{v}}$. Then, Proposition 3.20 and the maximality of $\mathfrak{s}$ in the set of paths $\left\{\mathfrak{v} \in \widehat{B}_{k+1}^{(\lambda,l)} \mid \mathfrak{v}^{(k-1)} = (\lambda, l-1)\right\}$ give

$$\gamma_{\sigma\to\lambda} m_{\mathfrak{u}} = m_{\mathfrak{s}} e_k = f_{\mathfrak{s}} e_k + \sum_{\mathfrak{v}\succ\mathfrak{s}} r_{\mathfrak{v}} f_{\mathfrak{v}} e_k = f_{\mathfrak{s}} e_k.$$

For the second equality, we evaluate $f_{\mathfrak{s}} e_k F_{\mathfrak{u}} e_k$ in two ways. On the one hand,

$$f_{\mathfrak{s}} e_k F_{\mathfrak{u}} e_k = \gamma_{\sigma\to\lambda} m_{\mathfrak{u}} F_{\mathfrak{u}} e_k = \gamma_{\sigma\to\lambda} f_{\mathfrak{u}} e_k.$$

On the other hand,

$$f_{\mathfrak{s}} e_k F_{\mathfrak{u}} e_k = e_k(\mathfrak{u},\mathfrak{u}) f_{\mathfrak{s}} F_{\mathfrak{u}''} e_k = e_k(\mathfrak{u},\mathfrak{u}) f_{\mathfrak{s}} e_k = e_k(\mathfrak{u},\mathfrak{u}) \gamma_{\sigma\to\lambda} m_{\mathfrak{u}},$$

where we used Corollary 3.10 and Corollary 3.22. Comparison of the two expressions gives the result.

(2) Let $\mu = \lambda \cup \{(j, \mu_j)\}$, $a' = \sum_{r=0}^{j} \mu_r$, and $a = a' + 2l - 2$. Let

$$u = u^{(k+1)}_{(\mu,l-1)\to(\lambda,l)} = w_{a',k-2l+2} e_k^{(l)}.$$

Following the proof of Lemma 5.1, we have

$$\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle \langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle^{-1} c_{(\lambda,l)} z_{(\lambda,l)} = u^* c_{(\mu,l-1)} z_{(\mu,l-1)} u z_{(\lambda,l)}. \tag{6.4}$$

In the following, let $\mathfrak{v} \in \widehat{B}_{k+1}^{(\lambda,l)}$ be defined by the condition $\mathfrak{v}_{\downarrow k} = \mathfrak{t}^{(\mu,l-1)}$. We consider two cases:

CASE 1: $l - 1 = 0$. Then $d_{\mathfrak{t}^{(\mu,0)}} = 1$ and $a = a'$, so

$$c_{(\mu,0)} z_{(\mu,0)} = c_{(\mu,0)} d_{\mathfrak{t}^{(\mu,0)}} z_{(\mu,0)} = c_{(\mu,0)} d_{\mathfrak{t}^{(\mu,0)}} F_{\mathfrak{t}^{(\mu,0)}},$$

using Lemma 3.7 part (3). Thus

$$\begin{aligned}\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle \langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle^{-1} c_{(\lambda,1)} z_{(\lambda,1)} &= u^* c_{(\mu,0)} d_{\mathfrak{t}^{(\mu,0)}} F_{\mathfrak{t}^{(\mu,0)}} w_{a,k} e_k z_{(\lambda,1)} \\ &= c_{(\lambda,l)} d_{\mathfrak{v}} F_{\mathfrak{v}} w_{a,k} e_k z_{(\lambda,1)}.\end{aligned} \tag{6.5}$$

CASE 2: $l - 1 \geqslant 1$. Then $d_{\mathfrak{t}^{(\mu,l-1)}} = e_{k-1}^{(l-1)} \cdots e_{2l-3}^{(l-1)}$, and

$$c_{(\mu,l-1)} = c_{(\mu,l-1)} d_{\mathfrak{t}^{(\mu,l-1)}} e_{2l-2}^{(l-1)} \cdots e_{k-1}^{(l-1)}.$$

Thus,

$$\begin{aligned}&\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle \langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle^{-1} c_{(\lambda,l)} z_{(\lambda,l)} \\ &\quad = u^* c_{(\mu,l-1)} d_{\mathfrak{t}^{(\mu,l-1)}} z_{(\mu,l-1)} e_{2l-2}^{(l-1)} \cdots e_{k-1}^{(l-1)} w_{a',k-2l+2} e_k^{(l)} z_{(\lambda,l)}\end{aligned} \tag{6.6}$$

Because $w_{a',k-2l+2}$ is a permutation of $\{2, \ldots, k-2l+2\}$, it follows from Lemma 2.18 part (3) that

$$e_{2l-2}^{(l-1)} \cdots e_{k-1}^{(l-1)} w_{a',k-2l+2} = w_{a,k} e_{2l-2}^{(l-1)} \cdots e_{k-1}^{(l-1)}.$$

There are at least two factors in the product $e_{2l-2}^{(l-1)} \cdots e_{k-1}^{(l-1)}$, since $k - 2l + 2 = |\mu| \geqslant 2$, so using Lemma 2.17,

$$e_{2l-2}^{(l-1)} \cdots e_{k-1}^{(l-1)} e_k^{(l)} = e_{2l-2}^{(l-1)} \cdots e_{k-2}^{(l-1)} e_k = e_k e_{2l-2}^{(l-1)} \cdots e_{k-2}^{(l-1)}.$$

Combining the last two displayed equations with (6.6), and also using Lemma 3.7 part (3), we get

$$\begin{aligned}&\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle \langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle^{-1} c_{(\lambda,l)} z_{(\lambda,l)} \\ &\quad = u^* c_{(\mu,l-1)} d_{\mathfrak{t}^{(\mu,l-1)}} F_{\mathfrak{t}^{(\mu,l-1)}} w_{a,k} e_k e_{2l-2}^{(l-1)} \cdots e_{k-2}^{(l-1)} z_{(\lambda,l)} \\ &\quad = c_{(\lambda,l)} d_{\mathfrak{v}} F_{\mathfrak{v}} w_{a,k} e_k e_{2l-2}^{(l-1)} \cdots e_{k-2}^{(l-1)} z_{(\lambda,l)}.\end{aligned} \tag{6.7}$$

If we adopt the convention that $e^{(l-1)}_{2l-2}\cdots e^{(l-1)}_{k-2}=1$ if $l-1=0$, then formula (6.7) is also valid in case 1, so we can treat both cases together.

By Lemma 5.3, we may write

$$f_{\mathfrak{v}}w_{a,k}=f_{\mathfrak{t}}+\sum_{\mathfrak{z}}r_{\mathfrak{z}}f_{\mathfrak{z}},$$

where the sum is over $\mathfrak{z}\in\widehat{B}^{(\lambda,l)}_{k+1}$ such that $\mathrm{Shape}(\mathfrak{z}\downarrow_{k-1})\neq(\lambda,l-1)$. Using the embedding of the cell module in B_{k+1} as in Lemma 3.7, as well as Proposition 3.20, we get

$$\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle^{-1}c_{(\lambda,l)}z_{(\lambda,l)}\quad=c_{(\lambda,l)}d_{\mathfrak{t}}F_{\mathfrak{t}}e_ke^{(l-1)}_{2l-2}\cdots e^{(l-1)}_{k-2}z_{(\lambda,l)}.$$

Applying Lemma 3.24 and point (1) above,

$$\begin{aligned}\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle^{-1}c_{(\lambda,l)}z_{(\lambda,l)}&=\\&=\frac{e_k(\mathfrak{s},\mathfrak{t})}{e_k(\mathfrak{s},\mathfrak{s})}c_{(\lambda,l)}d_{\mathfrak{s}}F_{\mathfrak{s}}e_ke^{(l-1)}_{2l-2}\cdots e^{(l-1)}_{k-2}z_{(\lambda,l)}\\&=\frac{e_k(\mathfrak{s},\mathfrak{t})}{e_k(\mathfrak{s},\mathfrak{s})}\gamma_{\sigma\to\lambda}c_{(\lambda,l)}d_{\mathfrak{u}}e^{(l-1)}_{2l-2}\cdots e^{(l-1)}_{k-2}z_{(\lambda,l)}.\end{aligned}\tag{6.8}$$

In case 1, $d_{\mathfrak{u}}e^{(l-1)}_{2l-2}\cdots e^{(l-1)}_{k-2}=1$, while in case 2, $d_{\mathfrak{u}}=e^{(l-1)}_{k-1}\cdots e^{(l-1)}_{2l-3}$, and

$$c_{(\lambda,l)}d_{\mathfrak{u}}e^{(l-1)}_{2l-2}\cdots e^{(l-1)}_{k-2}=c_{(\lambda,l)}.$$

Thus, in both cases we get

$$\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle^{-1}c_{(\lambda,l)}z_{(\lambda,l)}=\frac{e_k(\mathfrak{s},\mathfrak{t})}{e_k(\mathfrak{s},\mathfrak{s})}\gamma_{\sigma\to\lambda}c_{(\lambda,l)}z_{(\lambda,l)}.$$

By Theorem 4.9, we have

$$\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle\langle f_{\mathfrak{t}'},f_{\mathfrak{t}'}\rangle^{-1}=e_k(\mathfrak{t},\mathfrak{t})\gamma_{\lambda\to\mu}.$$

Hence we obtain the stated formula for $e_k(\mathfrak{s},\mathfrak{t})$. The formula for $e_k(\mathfrak{t},\mathfrak{s})$ can now be obtained from the relation $e_k(\mathfrak{s},\mathfrak{t})e_k(\mathfrak{t},\mathfrak{s})=e_k(\mathfrak{s},\mathfrak{s})e_k(\mathfrak{t},\mathfrak{t})$.

(3) Let $\mathfrak{t}\in\widehat{B}^{(\lambda,l)}_{k+1}$, where $\mathfrak{t}\overset{k}{\sim}\mathfrak{s}$ and $\mathfrak{t}^{(k)}=(\mu,l)$. By part (1), $f_{\mathfrak{s}}e_k=\gamma_{\sigma\to\lambda}m_{\mathfrak{u}}$ and hence by the orthogonality of the set of $f_{\mathfrak{v}}$,

$$\gamma_{\sigma\to\lambda}\langle m_{\mathfrak{u}},f_{\mathfrak{t}}\rangle=\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle e_k(\mathfrak{t},\mathfrak{s}).\tag{6.9}$$

To compute the bilinear form $\langle m_{\mathfrak{u}},f_{\mathfrak{t}}\rangle$, it will be convenient to work with the elements $m_{\mathfrak{u}\mathfrak{u}}$ and $m_{\mathfrak{u}\mathfrak{t}}$, and we begin by computing these elements.

We have $d_{\mathfrak{u}}=e^{(l-1)}_{k-1}e^{(l-1)}_{k-2}d_{\mathfrak{t}(\lambda,l-1)}$. From Lemma 5.5 part (2), we have $m_{\mathfrak{u}\,\mathfrak{u}}=m_{(\lambda,l-1)}e_k$. Write $\lambda=\mu\cup\{(j,\lambda_j)\}$, $a'=\sum_{r=1}^{j}\lambda_r$ and $a=2l-2+a'$. We have

$$d^{(k+1)}_{(\mu,l)\to(\lambda,l)}=d^{(k+1-2l)}_{\mu\to\lambda}e^{(l)}_{k-1}=w_{a',k+1-2l}e^{(l)}_{k-1}$$

and

$$d^{(k)}_{(\lambda,l-1)\to(\mu,l)}=u^{(k+1-2l)}_{\mu\to\lambda}e^{(l-1)}_{k-2}=w_{k+1-2l,a'}\sum_{r=0}^{\mu_j}w_{a',a'-r}e^{(l-1)}_{k-2}.$$

Thus,

$$d_{\mathfrak{t}}=d^{(k+1-2l)}_{\mu\to\lambda}e^{(l)}_{k-1}u^{(k+1-2l)}_{\mu\to\lambda}e^{(l-1)}_{k-2}d_{\mathfrak{t}(\lambda,l-1)}.$$

We consider two cases:

CASE 1: $l-1=0$. We have $a'=a$, $d_{\mathfrak{u}}=1$, and $d_{\mathfrak{t}}=w_{a,k-1}e_{k-1}w_{k-1,a}\sum_{r=0}^{\mu_j}w_{a,a-r}$. Thus

$$m_{\mathfrak{u}\,\mathfrak{t}}=c_{(\lambda,0)}e_kd_{\mathfrak{t}}=m_{(\lambda,0)}e_kw_{a,k-1}e_{k-1}w_{k-1,a}\sum_{r=0}^{\mu_j}w_{a,a-r}. \tag{6.10}$$

CASE 2: $l-1\geqslant 1$. Then $d_{\mathfrak{t}^{(\lambda,l-1)}}=e_{k-3}^{(l-1)}\cdots e_{2l-3}^{(l-1)}$, with $k-2l+1=|\lambda|\geqslant 1$ factors. We compute $m_{\mathfrak{u}\,\mathfrak{t}}$:

$$\begin{aligned} m_{\mathfrak{u}\,\mathfrak{t}}&=d_{\mathfrak{u}}c_{(\lambda,0)}e_k^{(l)}d_{\mu\to\lambda}^{(k+1-2l)}e_{k-1}^{(l)}u_{\mu\to\lambda}^{(k+1-2l)}e_{k-2}^{(l-1)}e_{k-3}^{(l-1)}\cdots e_{2l-3}^{(l-1)}\\ &=d_{\mathfrak{u}}c_{(\lambda,0)}e_k^{(l)}d_{\mu\to\lambda}^{(k+1-2l)}e_{k-1}^{(l)}e_{k-2}^{(l-1)}e_{k-3}^{(l-1)}\cdots e_{2l-3}^{(l-1)}u_{\mu\to\lambda}^{(k+1-2l)}[2l-2]\\ &=d_{\mathfrak{u}}c_{(\lambda,0)}e_kd_{\mu\to\lambda}^{(k+1-2l)}e_{k-2}^{(l-1)}e_{k-3}^{(l-1)}\cdots e_{2l-3}^{(l-1)}e_{k-1}u_{\mu\to\lambda}^{(k+1-2l)}[2l-2]\\ &=d_{\mathfrak{u}}c_{(\lambda,0)}e_ke_{k-2}^{(l-1)}e_{k-3}^{(l-1)}\cdots e_{2l-3}^{(l-1)}d_{\mu\to\lambda}^{(k+1-2l)}[2l-2]e_{k-1}u_{\mu\to\lambda}^{(k+1-2l)}[2l-2]\\ &=m_{(\lambda,l-1)}e_kd_{\mu\to\lambda}^{(k+1-2l)}[2l-2]e_{k-1}u_{\mu\to\lambda}^{(k+1-2l)}[2l-2]\\ &=m_{(\lambda,l-1)}e_kw_{a,k-1}e_{k-1}w_{k-1,a}\sum_{r=0}^{\mu_j}w_{a,a-r}, \end{aligned} \tag{6.11}$$

where we have used Lemma 2.18 part (3) in lines (2) and (4), commutation relations in line (3), and contraction relations in line (5). The final expresssion is valid in both cases 1 and 2.

Now that we have expressions for $m_{\mathfrak{u}\,\mathfrak{u}}$ and $m_{\mathfrak{u}\,\mathfrak{t}}$, we are ready to compute $\langle m_{\mathfrak{u}},f_{\mathfrak{t}}\rangle$. By a variant on Lemma 3.7 part (2),

$$\begin{aligned} \langle m_{\mathfrak{u}},f_{\mathfrak{t}}\rangle m_{\mathfrak{u}\,\mathfrak{u}}z_{(\lambda,l)}&=m_{\mathfrak{u}\,\mathfrak{t}}F_{\mathfrak{t}}m_{\mathfrak{u}\,\mathfrak{u}}\\ &=m_{(\lambda,l-1)}e_kw_{a,k-1}e_{k-1}w_{k-1,a}\sum_{r=0}^{j}w_{a,a-r}F_{\mathfrak{t}}m_{(\lambda,l-1)}e_k. \end{aligned} \tag{6.12}$$

Because $F_{\mathfrak{t}}$ commutes with $m_{(\lambda,l-1)}$ by Lemma 3.7 part (4), and $(\sum_{r=0}^{j}w_{a,a-r})m_{(\lambda,l-1)}=\lambda_im_{(\lambda,l-1)}$, the last expression reduces to

$$\lambda_im_{(\lambda,l-1)}e_kw_{a,k-1}e_{k-1}w_{k-1,a}F_{\mathfrak{t}}m_{(\lambda,l-1)}e_k. \tag{6.13}$$

In this last expression, we will rewrite F_t as $\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle^{-1}F_{\mathfrak{t}\,\mathfrak{t}}$ and then use that $f_{\mathfrak{x}}^*\mapsto F_{\mathfrak{x}\,\mathfrak{t}}$ determines an isomorphism from the opposite cell module $(\Delta_{k+1,\mathbb{F}}^{(\lambda,l)})^*$ to $\operatorname{span}_{\mathbb{F}}\{F_{\mathfrak{x}\,\mathfrak{t}}\}$, by Lemma 3.7 part (1). By Lemma 5.3, the path $\mathfrak{v}=\mathfrak{t}w_{a,k-1}$ is defined in $\widehat{B}_{k+1}^{(\lambda,l)}$, and

$$\mathfrak{v}=(\cdots,(\mu,l-1),(\lambda,l-1),(\mu,l),(\lambda,l)).$$

Furthermore, we may write

$$f_{\mathfrak{t}}w_{a,k-1}=f_{\mathfrak{v}}+\sum_{\mathfrak{z}\succ\mathfrak{v}}r_{\mathfrak{z}}f_{\mathfrak{z}},$$

where the sum is over $\mathfrak{z}\in\widehat{B}_{k+1}^{(\lambda,l)}$ such that $\mathfrak{z}^{(k-2)}\neq(\mu,l-1)$. It follows that

$$\begin{aligned} e_ke_{k-1}w_{k-1,a}F_{\mathfrak{t}}&=\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle^{-1}\,e_ke_{k-1}w_{k-1,a}F_{\mathfrak{t}\,\mathfrak{t}}\\ &=\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle^{-1}e_ke_{k-1}(F_{\mathfrak{v}\,\mathfrak{t}}+\sum_{\mathfrak{z}\succ\mathfrak{v}}r_{\mathfrak{z}}F_{\mathfrak{z}\,\mathfrak{t}})\\ &=\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle^{-1}e_ke_{k-1}F_{\mathfrak{v}\,\mathfrak{t}}=\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle^{-1}e_{k-1}(\mathfrak{v},\mathfrak{v})e_kF_{\mathfrak{v}\,\mathfrak{t}}, \end{aligned} \tag{6.14}$$

where we have used Proposition 3.20 both for e_{k-1} and for e_k. Substituting this in (6.13), we get

$$\langle f_{\mathfrak{t}},f_{\mathfrak{t}}\rangle^{-1}e_{k-1}(\mathfrak{v},\mathfrak{v})\lambda_im_{(\lambda,l-1)}e_kw_{a,k-1}F_{\mathfrak{v}\,\mathfrak{t}}m_{(\lambda,l-1)}e_k. \tag{6.15}$$

Now use that $F_{\mathfrak{v}\mathfrak{t}} = z_{(\lambda,l)}z_{(\mu,l)}z_{(\lambda,l-1)}F_{\mathfrak{v}\mathfrak{t}}$, that $z_{(\lambda,l)}z_{(\mu,l)}z_{(\lambda,l-1)}$ commutes with $w_{a-2,k-1}$, and that $m_{(\lambda,l-1)}z_{(\lambda,l)}z_{(\mu,l)}z_{(\lambda,l-1)} = m_{(\lambda,l-1)}F_{\mathfrak{t}}$ by Lemma 3.7 part (4) to write (6.15) as

$$\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle^{-1}e_{k-1}(\mathfrak{v},\mathfrak{v})\lambda_i e_k m_{(\lambda,l-1)}F_{\mathfrak{t}}w_{a,k-1}F_{\mathfrak{v}\mathfrak{t}}m_{(\lambda,l-1)}e_k. \tag{6.16}$$

Using Corollary 5.6, $F_{\mathfrak{t}}w_{a,k-1}F_{\mathfrak{v}\mathfrak{t}} = \gamma_{\mu\to\lambda}\lambda_i^{-1}F_{\mathfrak{t}\mathfrak{t}}$. Substituting this in (6.16) and again using $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle^{-1}F_{\mathfrak{t}\mathfrak{t}} = F_{\mathfrak{t}}$ and $F_{\mathfrak{t}} = F_{\mathfrak{t}}z_{(\lambda,l)}$ yields

$$e_{k-1}(\mathfrak{v},\mathfrak{v})\gamma_{\mu\to\lambda}m_{(\lambda,l-1)}e_kF_{\mathfrak{t}}e_km_{(\lambda,l-1)}z_{(\lambda,l)}. \tag{6.17}$$

Now use $e_kF_{\mathfrak{t}}e_k = e_k(\mathfrak{t},\mathfrak{t})F_{\mathfrak{t}''}e_k$ (where $\mathfrak{t}'' = \mathfrak{t}_{\downarrow k-1} = \mathfrak{t}^{(\lambda,l-1)}$) from Corollary 3.22, and $m_{(\lambda,l-1)}F_{\mathfrak{t}''}m_{(\lambda,l-1)} = \langle f_{\mathfrak{t}''}, f_{\mathfrak{t}''}\rangle m_{(\lambda,l-1)}z_{(\lambda,l-1)}$, from Lemma 3.7 part (2), to rewrite (6.17) as

$$e_k(\mathfrak{t},\mathfrak{t})e_{k-1}(\mathfrak{v},\mathfrak{v})\gamma_{\mu\to\lambda}\langle f_{\mathfrak{t}''}, f_{\mathfrak{t}''}\rangle m_{(\lambda,l-1)}e_kz_{(\lambda,l-1)}z_{(\lambda,l)}. \tag{6.18}$$

It follows from Proposition 3.20 that $e_kz_{(\lambda,l-1)}z_{(\lambda,l)} = e_kz_{(\lambda,l)}$, and we know that $m_{(\lambda,l-1)}e_k = m_{\mathfrak{u}\mathfrak{u}}$. Moreover, $e_{k-1}(\mathfrak{v},\mathfrak{v})\gamma_{\mu\to\lambda} = \gamma^{(k+1)}_{(\lambda,l-1)\to(\mu,l)}$, so $e_{k-1}(\mathfrak{v},\mathfrak{v})\gamma_{\mu\to\lambda}\langle f_{\mathfrak{t}''}, f_{\mathfrak{t}''}\rangle = \langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle$, using Theorem 4.9. Thus (6.18) becomes

$$e_k(\mathfrak{t},\mathfrak{t})\langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle m_{\mathfrak{u}\mathfrak{u}}z_{(\lambda,l)}. \tag{6.19}$$

Comparing with our starting point (6.12), we arrive at

$$\langle m_{\mathfrak{u}}, f_{\mathfrak{t}}\rangle = e_k(\mathfrak{t},\mathfrak{t})\langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle. \tag{6.20}$$

Finally, combining (6.20) and (6.9), and applying Theorem 4.9 once more

$$e_k(\mathfrak{t},\mathfrak{s}) = \langle f_{\mathfrak{t}}, f_{\mathfrak{t}}\rangle^{-1}\gamma_{\sigma\to\lambda}e_k(\mathfrak{t},\mathfrak{t})\langle f_{\mathfrak{t}'}, f_{\mathfrak{t}'}\rangle = \frac{\gamma_{\sigma\to\lambda}}{\gamma_{\mu\to\lambda}}e_k(\mathfrak{t},\mathfrak{t}), \tag{6.21}$$

which is the desired formula for $e_k(\mathfrak{t},\mathfrak{s})$ The formula for $e_k(\mathfrak{s},\mathfrak{t})$ is now obtained from the relation $e_k(\mathfrak{s},\mathfrak{t})e_k(\mathfrak{t},\mathfrak{s}) = e_k(\mathfrak{s},\mathfrak{s})e_k(\mathfrak{t},\mathfrak{t})$. □

Proof of Theorem 4.11.

(1) This follows from Proposition 3.20.

(2) We know from Corollary 3.17 that the matrix coefficients $e_k(\mathfrak{s},\mathfrak{t})$ depend only on $\mathfrak{s}_{[k-1,k+1]}$ and $\mathfrak{t}_{[k-1,k+1]}$. The diagonal matrix entries $e_k(\mathfrak{t},\mathfrak{t})$ are determined by Nazarov's formula (4.1). If $\lambda = \emptyset$, there are no off–diagonal matrix entries to be determined, so we assume that $\lambda \neq \emptyset$. Denote $(\sigma,l) = \max\{(v,r) \in \widehat{B}_k \mid (v,r) \to (\lambda,l)\}$ and $\mathfrak{v} \in \widehat{B}^{(\lambda,l)}_{k+1}$, where $\mathfrak{v} \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{v}^{(k)} = (\sigma,l)$. We have four cases to consider.

CASE 1. Assume that $\mathfrak{t}^{(k)} = (\mu,l-1)$ and $\mathfrak{s}^{(k)} = (\rho,l-1)$. By Proposition 6.1(2),

$$e_k(\mathfrak{v},\mathfrak{t}) = \frac{\gamma_{\lambda\to\mu}}{\gamma_{\sigma\to\lambda}}e_k(\mathfrak{v},\mathfrak{v})e_k(\mathfrak{t},\mathfrak{t}) \qquad \text{and} \qquad e_k(\mathfrak{s},\mathfrak{v}) = \frac{\gamma_{\sigma\to\lambda}}{\gamma_{\lambda\to\rho}}.$$

Therefore, the relation $e_k(\mathfrak{s},\mathfrak{v})e_k(\mathfrak{v},\mathfrak{t}) = e_k(\mathfrak{v},\mathfrak{v})e_k(\mathfrak{s},\mathfrak{t})$ yields the required expression for $e_k(\mathfrak{s},\mathfrak{t})$.

CASE 2. Assume that $\mathfrak{t}^{(k)} = (\mu,l)$ and $\mathfrak{s}^{(k)} = (\rho,l-1)$. By Proposition 6.1,

$$e_k(\mathfrak{v},\mathfrak{t}) = \frac{\gamma_{\mu\to\lambda}}{\gamma_{\sigma\to\lambda}}e_k(\mathfrak{v},\mathfrak{v}) \qquad \text{and} \qquad e_k(\mathfrak{s},\mathfrak{v}) = \frac{\gamma_{\sigma\to\lambda}}{\gamma_{\lambda\to\rho}}.$$

The relation $e_k(\mathfrak{s},\mathfrak{v})e_k(\mathfrak{v},\mathfrak{t}) = e_k(\mathfrak{v},\mathfrak{v})e_k(\mathfrak{s},\mathfrak{t})$ now yields the required expression for $e_k(\mathfrak{s},\mathfrak{t})$.

CASE 3. Assume that $\mathfrak{t}^{(k)} = (\mu,l-1)$ and $\mathfrak{s}^{(k)} = (\rho,l)$. Using the second case above, we obtain $e_k(\mathfrak{t},\mathfrak{s}) = \gamma_{\rho\to\lambda}\gamma^{-1}_{\lambda\to\mu}$. Hence the formula for $e_k(\mathfrak{s},\mathfrak{t})$ follows.

CASE 4. Assume that $\mathfrak{t}^{(k)} = (\mu,l)$ and $\mathfrak{s}^{(k)} = (\rho,l)$. By Proposition 6.1(3),

$$e_k(\mathfrak{v},\mathfrak{t}) = \frac{\gamma_{\mu\to\lambda}}{\gamma_{\sigma\to\lambda}}e_k(\mathfrak{v},\mathfrak{v}) \qquad \text{and} \qquad e_k(\mathfrak{s},\mathfrak{v}) = \frac{\gamma_{\sigma\to\lambda}}{\gamma_{\rho\to\lambda}}e_k(\mathfrak{s},\mathfrak{s}).$$

The relation $e_k(\mathfrak{s},\mathfrak{v})e_k(\mathfrak{v},\mathfrak{t}) = e_k(\mathfrak{v},\mathfrak{v})e_k(\mathfrak{s},\mathfrak{t})$ now yields the required expression for $e_k(\mathfrak{s},\mathfrak{t})$.

(3) We use the relation $s_kL_{k+1} - L_ks_k = 1 - e_k$ and the diagonal action of the Jucys–Murphy elements on the seminormal basis. □

7. The seminormal representations and tensor space

Let n be a non–zero integer. In this section we give an explicit description of the simple $B_f(n)$ modules which factor through $B_f(n)/\operatorname{rad}(\tau_n)$, where $\operatorname{rad}(\tau_n)$ is the radical of the Markov trace. When n is a positive integer (resp. an even negative integer) $B_f(n)/\operatorname{rad}(\tau_n)$ is isomorphic to the the centralizer algebra of the orthogonal group (resp. the symplectic group) acting on the f–fold tensor power of its vector representation.

We will be dealing with several specializations of the Brauer algebras simultaneously. We recall the notation $R = \mathbb{Z}[\boldsymbol{z}]$ and $\mathbb{F} = \mathbb{Q}(\boldsymbol{z})$. We write $\mathcal{B}_k = B_k(R,\boldsymbol{z})$ and $B_k(\boldsymbol{z}) = B_k(\mathbb{F},\boldsymbol{z})$.

Write $R_n = \mathbb{Z}[\boldsymbol{z}]_{(\boldsymbol{z}-n)}$ for the localization of $Z[\boldsymbol{z}]$ at the prime $z - n$; i.e. $R_n \subset \mathbb{F}$ is the set of rational functions with denominators not divisible by $z - n$. R_n is a discrete valuation ring with maximal ideal $(z - n)R_n$ and residue field $\mathbb{Q}$; the maximal ideal is the kernel of the evaluation homomorphism from R_n to $\mathbb{Q}$ determined by $z \mapsto n$.

Write $B_k(n)$ for $B_k(\mathbb{Q}, n)$. Then $B_k(n) \cong B_k(R_n,\boldsymbol{z}) \otimes_{R_n} \mathbb{Q}$; where R_n acts on $\mathbb{Q}$ by the evaluation homomorphism. The evaluation map from R_n to $\mathbb{Q}$ extends to an evaluation map $B_k(R_n,\boldsymbol{z}) \to B_k(n)$ given by $a \mapsto a \otimes 1$, or more concretely by $\sum_d f_d\, d \mapsto \sum_d f_d(n)\, d$, where the sum is over the basis of Brauer diagrams and $f_d \in R_n$. We will denote this map by $a \mapsto a(n)$. We will refer to $B_k(R_n,\boldsymbol{z}) \subset B_k(\mathbb{F},\boldsymbol{z})$ as the ring of *evaluable elements*.

Recall that τ denotes the Markov trace on $B_k(\boldsymbol{z})$. Let τ_n denote the Markov trace on $B_k(n)$. If $a \in B_k(\boldsymbol{z})$ is evaluable, then $\tau(a) \in R_n$ and $\tau_n(a(n)) = \tau(a)(n)$. The *radical* of the Markov trace τ_n is the set of $x \in B_k(n)$ such that for all $y \in B_k(n)$, $\tau(xy) = 0$; the radical of the trace, denoted $\operatorname{rad}(\tau_n)$ or $I_k(n)$, is a two sided ideal in $B_k(n)$. It is observed in [18, Lemma 3.1], that $I_k(n) \subseteq I_{k+1}(n)$.

For a non–zero integer n, a Young diagram μ is called *n–permissible* if for all Young diagrams $\lambda \subseteq \mu$, $P_\lambda(n) \neq 0$, where P_λ is the El Samra–King polyomial. The following are necessary and sufficient conditions for a Young diagram μ be n–permissible ([18], Corollary 3.5):

- If $n > 0$, $\tilde{\mu}_1 + \tilde{\mu}_2 \leqslant n$.
- If $n = -2k$ is even and negative, $\mu_1 \leqslant k$.
- If n is odd and negative, $\mu_1 + \mu_2 \leqslant 2 - n$.

We call an element $(\mu, m) \in \widehat{B}_f$ n–permissible if μ is n–permissible. A path $\mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ is called n–permissible if $\mathfrak{t}^{(j)}$ is n–permissible for all j.

Let $\mathfrak{s},\mathfrak{t}$ be n–permissible paths. We will show that $\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$ is a unit in R_n, and that the matrix entries $e_i(\mathfrak{s},\mathfrak{t})$ and $s_i(\mathfrak{s},\mathfrak{t})$ are in R_n. We will also show that the idempotents $F_\mathfrak{t}$ and the basis elements $F_{\mathfrak{s}\mathfrak{t}}$ are evaluable.

Lemma 7.1. *Let $\mathfrak{s},\mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ be a n–permissible paths. Then*

(1) For all $i < f$, if $e_i(\mathfrak{s},\mathfrak{t}) \neq 0$, then $e_i(\mathfrak{s},\mathfrak{t})$ is a unit in R_n.
(2) $\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$ is a unit in R_n.

Proof. When $e_i(\mathfrak{t},\mathfrak{t}) \neq 0$, it is the ratio of El Samra–King polynomials of two n–permissible Young diagrams by Corollary 3.22 part (2), hence $e_i(\mathfrak{t},\mathfrak{t})$ is a unit in R_n. For $\mathfrak{s} \neq \mathfrak{t}$, statement (1) follows from Theorem 4.11 part (2).

Statement (2) follows from Definition 4.7 and the recursive formula (4.8) for $\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$. □

Remark 7.2. Let $\mathfrak{s}, \mathfrak{t} \in \widehat{B}^{(\lambda,l)}$ be n–permissible. Then $F_\mathfrak{t}$ is evaluable if and only if $F_{\mathfrak{t}\mathfrak{t}}$ is evaluable, because $F_{\mathfrak{t}\mathfrak{t}} = \langle f_\mathfrak{t}, f_\mathfrak{t}\rangle F_\mathfrak{t}$, and $\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle$ is a unit in R_n. If $F_\mathfrak{s}$ and $F_\mathfrak{t}$ are evaluable, then so is $F_{\mathfrak{s}\mathfrak{t}} = F_\mathfrak{s} m_{\mathfrak{s}\mathfrak{t}} F_\mathfrak{t}$. Conversely, if $F_{\mathfrak{s}\mathfrak{t}}$ is evaluable, so are $F_{\mathfrak{t}\mathfrak{s}} = F^*_{\mathfrak{s}\mathfrak{t}}$ and $F_{\mathfrak{t}\mathfrak{t}} = \langle f_\mathfrak{s}, f_\mathfrak{s}\rangle^{-1} F_{\mathfrak{t}\mathfrak{s}} F_{\mathfrak{s}\mathfrak{t}}$.

Following [13, Section 4], say two paths $\mathfrak{s}, \mathfrak{t} \in \widehat{B}^{(\cdot)}_k$ are *residue equivalent*, and write $\mathfrak{s} \approx \mathfrak{t}$ if $c_\mathfrak{s}(j) \equiv c_\mathfrak{t}(j) \mod (\boldsymbol{z} - n)$ for all j.

Lemma 7.3. *Let $\mathfrak{t} \in \widehat{B}^{(\cdot)}_{f+s}$ and write $\mathfrak{t}_0 = \mathfrak{t}_{\downarrow f}$. Suppose that $F_{\mathfrak{t}_0}$ is evaluable. Let $[\mathfrak{t}]$ be the set of $\mathfrak{s} \in \widehat{B}^{(\cdot)}_{f+s}$ such that $\mathfrak{s}_{\downarrow f} = \mathfrak{t}_0$ and and $\mathfrak{s} \approx \mathfrak{t}$. Then $F_{[\mathfrak{t}]} := \sum_{\mathfrak{s}\in[\mathfrak{t}]} F_\mathfrak{s}$ is an evaluable idempotent.*

Proof. The proof of [13, Lemma 4.2] applies with minor changes. □

Lemma 7.4. *Let $\mathfrak{t} \in \widehat{B}^{(\cdot)}_{f+1}$. There is at most one path $\mathfrak{s} \in \widehat{B}^{(\cdot)}_{f+1}$ such that $\mathfrak{s} \neq \mathfrak{t}$, $\mathfrak{s}' = \mathfrak{t}'$ and $\mathfrak{s} \approx \mathfrak{t}$.*

Proof. Two edges $c((\mu, m) \to (\lambda, l))$ and $c((\mu, m) \to (\lambda', l'))$ in $\widehat{B}$ are congruent modulo $\boldsymbol{z} - n$ only if one of the two edge involves adding a cell α to μ and the other removing a cell β from μ. Moreover the condition for the the contents to be congruent modulo $\boldsymbol{z} - n$ is $c(\alpha) + c(\beta) = 1 - n$. Therefore, for a given edge $(\mu, m) \to (\lambda, l)$, there is at most one other edge $(\mu, m) \to (\lambda', l')$ such that the contents of the edges are congruent modulo $\boldsymbol{z} - n$. □

Lemma 7.5. *Let $f \geqslant 1$ and $\mathfrak{t} \in \widehat{B}^{(\cdot)}_f$. Suppose that $\mathfrak{t}' = \mathfrak{t}_{\downarrow f-1}$ is n–permissble. Then $F_\mathfrak{t}$ is evaluable.*

Proof. The proof is by induction on f. The base case $f = 1$ is obvious. Suppose that $f > 1$, and that $\mathfrak{t} \in \widehat{B}^{(\cdot)}_f$ such that $\mathfrak{t}'$ is n–permissible. Write $(\mu, m) = \mathfrak{t}^{(f-1)}$. By the appropriate induction hypothesis, $F_{\mathfrak{t}'}$ is evaluable.

By Corollary 3.12, $F_\mathfrak{t}$ is evaluable unless there exists a path $\mathfrak{s}$ such that $\mathfrak{s} \neq \mathfrak{t}$, $\mathfrak{s}' = \mathfrak{t}'$, and $c_\mathfrak{s}(f) \equiv c_\mathfrak{t}(f) \mod (\boldsymbol{z} - n)$. There is at most one such path by Lemma 7.4, and if such a path $\mathfrak{s}$ exists then $F_\mathfrak{s} + F_\mathfrak{t}$ is evaluable by Lemma 7.3. Thus $F_\mathfrak{t}$ is evaluable if and only if $F_\mathfrak{s}$ is evaluable. One of the two paths $\mathfrak{t}, \mathfrak{s}$ has the form

$$(\cdots, (\mu, m), (\mu_-, m + 1)),$$

and the other

$$(\cdots, (\mu, m), (\mu_+, m)),$$

where μ_- is obtained by removing a cell from μ and μ_+ by adding a cell. We can assume without loss of generality that $\mathfrak{t}$ is the first of these two paths, and in particular, that $\mathfrak{t}$ is n–permissible.

To prove that $F_\mathfrak{t}$ is evaluable, first consider an n–permissible path of the form

$$\mathfrak{u} = (\cdots, (\mu_-, m), (\mu, m), (\mu_-, m + 1)).$$

By the induction hypothesis, $F_{\mathfrak{u}'}$ is evaluable. We have $F_{\mathfrak{u}'} e_{f-1} F_{\mathfrak{u}'} = F_\mathfrak{u} e_{f-1} F_\mathfrak{u}$, using Proposition 3.20, so $F_\mathfrak{u} e_{f-1} F_\mathfrak{u}$ is evaluable. But $F_\mathfrak{u} = d^{-1} F_\mathfrak{u} e_{f-1} F_\mathfrak{u}$, where $d = e_{f-1}(\mathfrak{u}, \mathfrak{u}) = (P_\mu(\boldsymbol{z})/P_{\mu_-}(\boldsymbol{z}))$, using Corollary 3.22. Hence $F_\mathfrak{u}$ is evaluable, since d is a unit in R_n. Finally, using Lemma 3.7 (1) and Corollary 3.13, we have

$$F_{\mathfrak{t}'\mathfrak{u}'} F_{\mathfrak{u}\mathfrak{u}} F_{\mathfrak{u}'\mathfrak{t}'} = \langle f_{\mathfrak{u}'}, f_{\mathfrak{u}'}\rangle^2 F_{\mathfrak{t}\mathfrak{t}} = \langle f_{\mathfrak{u}'}, f_{\mathfrak{u}'}\rangle^2 \langle f_\mathfrak{t}, f_\mathfrak{t}\rangle F_\mathfrak{t}$$

Using Lemma 7.1 and Remark 7.2, it follows that $F_\mathfrak{t}$ is evaluable. □

We will now construct a cellular basis $\{h_{\mathfrak{s}\mathfrak{t}}\}$ of $B_f(R_n, \boldsymbol{z})$ indexed by pairs of paths in $\widehat{B}^{(\cdot)}_f$ of the same shape, with the properties:

1. If both $\mathfrak{s}$ and $\mathfrak{t}$ are n–permissible, then $h_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}\mathfrak{t}}$.
2. The set of $h_{\mathfrak{s}\mathfrak{t}}(n)$ where at least one of $\mathfrak{s}$, $\mathfrak{t}$ is *not* n–permissible is a basis of $\mathrm{rad}(\tau_n)$.

The construction is a variant of the construction of the basis $g_{\mathfrak{s}\,\mathfrak{t}}$ in [13, Theorem 4.5].

If $\mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ is n–permissible, set $[\mathfrak{t}] = \{\mathfrak{t}\}$. If $\mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ is not n–permissible, let $(\mu, m) = \mathfrak{t}^{(k)}$ be the first non n–permissible point on the path $\mathfrak{t}$ and write $\mathfrak{t}_0 = \mathfrak{t}_{\downarrow k}$. According to Lemma 7.5, $F_{\mathfrak{t}_0}$ is evaluable. Let $[\mathfrak{t}] = \{\mathfrak{s} \in \widehat{B}_f^{(\cdot)} \mid \mathfrak{s}_{\downarrow k} = \mathfrak{t}_0 \text{ and } \mathfrak{s} \approx \mathfrak{t}\}$ and let $F_{[\mathfrak{t}]} = \sum_{\mathfrak{s} \in [\mathfrak{t}]} F_{\mathfrak{s}}$. Then the idempotent $F_{[\mathfrak{t}]}$ is evaluable by Lemma 7.3.

Definition 7.6. For $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)}$, let $h_{\mathfrak{s}\,\mathfrak{t}} = F_{[\mathfrak{s}]} m_{\mathfrak{s}\,\mathfrak{t}} F_{[\mathfrak{t}]}$ and let $h_{\mathfrak{t}} \in \Delta_{k,R_n}^{(\lambda,l)}$ be defined by $h_{\mathfrak{t}} = m_{\mathfrak{t}} F_{[\mathfrak{t}]}$.

Lemma 7.7.

(1) Let $(\lambda, l) \in \widehat{B}_f$ and $\mathfrak{t} \in \widehat{B}_f^{(\lambda,l)}$. There exist coefficients $\beta_{\mathfrak{s}} \in R_n$, for $\mathfrak{s} \in \widehat{B}_k^{(\lambda,l)}$, such that

$$h_{\mathfrak{t}} = m_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} \beta_{\mathfrak{s}} m_{\mathfrak{s}}. \tag{7.1}$$

(2) $\{h_{\mathfrak{t}} \mid \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)}\}$ is an R_n–basis of the cell module $\Delta_{f,R_n}^{(\lambda,l)}$ and

$$\{h_{\mathfrak{s}\,\mathfrak{t}} \mid (\lambda, l) \in \widehat{B}_f \text{ and } \mathfrak{s}, \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)}\}$$

is a cellular basis of $B_f(R_n, \boldsymbol{z})$.

Proof. For part (1), start with $m_{\mathfrak{t}} = f_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r'_{\mathfrak{s}} f_{\mathfrak{s}}$ (where the coefficients are in $\mathbb{F}$) and multiply with $F_{[\mathfrak{t}]}$ on the right. This gives $h_{\mathfrak{t}} = f_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t},\, \mathfrak{s} \in [\mathfrak{t}]} r'_{\mathfrak{s}} f_{\mathfrak{s}}$. By applying (3.2), we obtain (7.1), but with coefficients *a priori* in $\mathbb{F}$. On the other hand, since $F_{[\mathfrak{t}]} \in B_f(R_n, \boldsymbol{z})$, we have $h_{\mathfrak{t}} = m_{\mathfrak{t}} F_{[\mathfrak{t}]} = \sum_{\mathfrak{s}} \gamma_{\mathfrak{s}} m_{\mathfrak{s}}$ with coefficients in R_n. Matching coefficients gives the the result. It follows that $\{h_{\mathfrak{t}} \mid \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)}\}$ is an R_n–basis of of the cell module $\Delta_{f,R_n}^{(\lambda,l)}$.

For the moment, write $A = B_f(R_n, \boldsymbol{z})$ for the sake of concision. With

$$\alpha : A^{\trianglerighteq(\lambda,l)}/A^{\triangleright(\lambda,l)} \to (\Delta_{f,R_n}^{(\lambda,l)})^* \otimes_{R_n} \Delta_{f,R_n}^{(\lambda,l)}$$

the A–A bimodule isomorphism determined by $m_{\mathfrak{s}\,\mathfrak{t}} + A^{\triangleright(\lambda,l)} \mapsto (m_{\mathfrak{s}})^* \otimes m_{\mathfrak{t}}$, we have $\alpha(h_{\mathfrak{s}\,\mathfrak{t}} + A^{\triangleright(\lambda,l)}) = (h_{\mathfrak{s}})^* \otimes h_{\mathfrak{t}}$. It follows from [7, Lemma 2.3] that $\{h_{\mathfrak{s}\,\mathfrak{t}}\}$ is a cellular basis of $B_f(R_n, \boldsymbol{z})$. ☐

Lemma 7.8. *For $f \geqslant 1$, the set of $h_{\mathfrak{s},\mathfrak{t}}(n)$ where $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ and at least one of $\mathfrak{s}, \mathfrak{t}$ is not n–permissible is a $\mathbb{Q}$–basis of $\mathrm{rad}(\tau_n) \subseteq B_f(n)$.*

Proof. Let M denote the span of the set of $h_{\mathfrak{s},\mathfrak{t}}(n)$ where $\mathfrak{s}, \mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ and at least one of $\mathfrak{s}, \mathfrak{t}$ is not n–permissible. We have to show that $M = \mathrm{rad}(\tau_n)$.

For any k, let $I_k(n)$ denote the radical of τ_n on $B_k(n)$.

First consider some $(\mu, m) \in \widehat{B}_k$ for $k \leqslant f$, with μ not n–permissible. Let $\mathfrak{t}_0 \in \widehat{B}_{k_0}^{(\mu,m)}$ be a path with $\mathfrak{t}'_0$ n–permissible. Then $F_{\mathfrak{t}_0}$ is evaluable by Lemma 7.5, and $\tau_n(F_{\mathfrak{t}_0}(n)) = P_\mu(n)/n^k = 0$. Moreover, $F_{\mathfrak{t}_0}(n) B_k(n) F_{\mathfrak{t}_0}(n) = \mathbb{Q} F_{\mathfrak{t}_0}(n)$, and it follows that $F_{\mathfrak{t}_0}(n) \in I_k(n)$.

Now let $\mathfrak{t} \in \widehat{B}_f^{(\cdot)}$ be a non n–permissible path. Let $(\mu, m) \in \widehat{B}_k$ and $\mathfrak{t}_0$ be as in the definition of $[\mathfrak{t}]$ preceding Definition 7.6. Then $F_{[\mathfrak{t}]} = F_{\mathfrak{t}_0} F_{[\mathfrak{t}]}$. Since $F_{\mathfrak{t}_0}(n) \in I_k(n)$ and $I_k(n) \subseteq I_f(n)$, we have $F_{[\mathfrak{t}]}(n) \in I_f(n)$. Finally if either $\mathfrak{s}$ or $\mathfrak{t}$ is not n permissible, then $h_{\mathfrak{s}\,\mathfrak{t}}(n) = F_{[\mathfrak{s}]}(n) m_{\mathfrak{s}\,\mathfrak{t}}(n) F_{[\mathfrak{t}]}(n) \in I_f(n)$. Thus we have $M \subseteq \mathrm{rad}(\tau_n)$.

If both $\mathfrak{s}$ and $\mathfrak{t}$ are n–permissible paths in $\widehat{B}_f^{(\lambda,l)}$, then $h_{\mathfrak{s}\,\mathfrak{t}} = F_{\mathfrak{s}\,\mathfrak{t}}$ and

$$\tau_n(F_{\mathfrak{s}\,\mathfrak{t}}(n) F_{\mathfrak{t}\,\mathfrak{s}}(n)) = \langle f_{\mathfrak{s}}, f_{\mathfrak{s}} \rangle(n) \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle(n) P_\lambda(n)/n^f \neq 0,$$

so $F_{\mathfrak{s}\,\mathfrak{t}}(n) \notin \operatorname{rad}(\tau_n)$. Finally, if $x \in \operatorname{rad}(\tau_n)$, let

$$x = \sum_{\mathfrak{s},\mathfrak{t}} \alpha(\mathfrak{s},\mathfrak{t}) F_{\mathfrak{s}\,\mathfrak{t}}(n) + \sum_{\mathfrak{u},\mathfrak{v}}{}' \beta(\mathfrak{u},\mathfrak{v}) h_{\mathfrak{u}\,\mathfrak{v}},$$

where the first sum is over pairs $(\mathfrak{s},\mathfrak{t})$ with both paths n–permissible and the second sum is over pairs of paths $(\mathfrak{u},\mathfrak{v})$ with at least one of the paths not n–permissible. For any pair $(\mathfrak{s},\mathfrak{t})$ with both paths n–permissible, we have $F_{\mathfrak{s}}(n) x F_{\mathfrak{t}}(n) = \alpha(\mathfrak{s},\mathfrak{t}) F_{\mathfrak{s}\,\mathfrak{t}}(n)$. The left side is in $\operatorname{rad}(\tau_n)$ and since $F_{\mathfrak{s}\,\mathfrak{t}}(n) \notin \operatorname{rad}(\tau_n)$, it follows that $\alpha(\mathfrak{s},\mathfrak{t}) = 0$. Thus we have shown that $\operatorname{rad}(\tau_n) \subseteq M$. □

For $x \in B_f(n)$, write $\overline{x}$ for the image of x in $B_f(n)/\operatorname{rad}(\tau_n)$. For $(\lambda, l) \in \widehat{B}_f$, let $a_{(\lambda,l)}$ denote the number of n–permissible paths of shape (λ, l).

The following theorem provides an explicit construction of the simple modules of $B_f(n)$ which factor through $B_f(n)/\operatorname{rad}(\tau_n)$. The result recovers Theorem 5.4.3 of [**?**].

Theorem 7.9.

(1) The set

$$\{\overline{F_{\mathfrak{s}\,\mathfrak{t}}(n)} \mid (\lambda, l) \in \widehat{B}_f, \mathfrak{s}, \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)} \text{ } n\text{–permissible}\}$$

is a $\mathbb{Q}$–basis of $B_f(n)/\operatorname{rad}(\tau_n)$.

(2) The set

$$\{\langle f_{\mathfrak{s}}, f_{\mathfrak{s}}\rangle(n)^{-1}\overline{F_{\mathfrak{s}\,\mathfrak{t}}(n)} \mid (\lambda, l) \in \widehat{B}_f, \mathfrak{s}, \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)} \text{ } n\text{–permissible}\}$$

is a system of matrix units and a $\mathbb{Q}$–basis of $B_f(n)/\operatorname{rad}(\tau_n)$. Thus,

$$B_f(n)/\operatorname{rad}(\tau_n) \cong \bigoplus_{(\lambda,l)} \operatorname{Mat}_{a_{(\lambda,l)}}(\mathbb{Q}),$$

where the sum is over $(\lambda, l) \in \widehat{B}_f$ with λ n–permissible.

(3) Let $(\lambda, l) \in \widehat{B}_f$ with λ n–permissible. For $\mathfrak{t} \in \widehat{B}_f^{(\lambda,l)}$ n–permissible, set $\bar{f}_{\mathfrak{t}} = \overline{F_{\mathfrak{t}^{(\lambda,l)}\,\mathfrak{t}}(n)}$. Then

$$V_{(\lambda,l)} = \operatorname{span}_{\mathbb{Q}}\{\bar{f}_{\mathfrak{t}} \mid \mathfrak{t} \in \widehat{B}_f^{(\lambda,l)} \text{ is } n\text{–permissible}\}$$

is a simple $B_f(n)$ module, with the module action $\bar{f}_{\mathfrak{t}} x = \bar{f}_{\mathfrak{t}} \overline{x}$ for $x \in B_f(n)$.

(4) For $\mathfrak{s}, \mathfrak{t}$ n–permissible paths of the same shape, the matrix coefficients $e_i(\mathfrak{s},\mathfrak{t})$ and $s_i(\mathfrak{s},\mathfrak{t})$ are in R_n.

(5) The structure constants of the generators e_i and s_i with respect to the basis $\{\bar{f}_{\mathfrak{t}}\}$ are obtained by evaluating the matrix entries $e_i(\mathfrak{s},\mathfrak{t})$ and $s_i(\mathfrak{s},\mathfrak{t})$ at $\boldsymbol{z} = n$:

$$\bar{f}_{\mathfrak{t}}\, e_i = \sum_{\mathfrak{s}} e_i(\mathfrak{s},\mathfrak{t})(n)\, \bar{f}_{\mathfrak{s}} \quad \textit{and} \quad \bar{f}_{\mathfrak{t}}\, s_i = \sum_{\mathfrak{s}} s_i(\mathfrak{s},\mathfrak{t})(n)\, \bar{f}_{\mathfrak{s}},$$

with the sums over n–permissible paths.

Proof. Point (1) follows from Lemmas 7.7 and 7.8. It follows from (1) and Proposition 3.6 part (7) that $\{\langle f_{\mathfrak{s}}, f_{\mathfrak{s}}\rangle(n)^{-1}\overline{F_{\mathfrak{s}\,\mathfrak{t}}(n)}\}$ is a system of matrix units and a basis of $B_f(n)/\operatorname{rad}(\tau_n)$, which proves (2). Point (3) is immediate from (2).

It was already shown in Lemma 7.1 that the matrix entries $e_i(\mathfrak{s},\mathfrak{t})$ for $\mathfrak{s}, \mathfrak{t}$ n–permissible are in R_n. For $\mathfrak{u}, \mathfrak{t}, \mathfrak{s}$ all n–permissible paths of the same shape, we have $F_{\mathfrak{u}\,\mathfrak{t}} s_i F_{\mathfrak{s}} = s_i(\mathfrak{s},\mathfrak{t}) F_{\mathfrak{u}\,\mathfrak{s}}$. Since the left side of this equation is evaluable, so is the right side, and since $F_{\mathfrak{u}\,\mathfrak{s}}(n) \neq 0$, this implies that $s_i(\mathfrak{s},\mathfrak{t}) \in R_n$. This proves point (4).

Finally, for point (5), if $\bar{f}_{\mathfrak{t}} e_i = \sum_{\mathfrak{s}} \alpha(\mathfrak{s}) \bar{f}_{\mathfrak{s}}$, then $\bar{f}_{\mathfrak{t}} e_i F_{\mathfrak{s}} = \alpha(\mathfrak{s}) \bar{f}_{\mathfrak{s}}$. This gives

$$\alpha(\mathfrak{s}) \overline{F_{\mathfrak{t}^{(\lambda,l)}\,\mathfrak{t}}(n)} = \overline{F_{\mathfrak{t}^{(\lambda,l)}\,\mathfrak{t}} e_i F_{\mathfrak{s}}(n)} = e_i(\mathfrak{s},\mathfrak{t})(n) \overline{F_{\mathfrak{t}^{(\lambda,l)}\,\mathfrak{t}}(n)},$$

so $\alpha(\mathfrak{s}) = e_i(\mathfrak{s}, \mathfrak{t})(n)$. The identical proof applies to the generators s_i. $\square$

Theorem 7.9 is equally valid over any field of characteristic zero, in particular over the complex numbers. If n is a positive integer, there is a well known homomorphism Φ from the Brauer algebra $B_f(\mathbb{C}, n)$ onto the centralizer of the complex orthogonal group $\mathrm{O(n, \mathbb{C})}$ acting on the f–fold tensor power of its vector representation. If n is an even negative integer, $n = -2k$, then there is a homomorphism Φ of $B_f(\mathbb{C}, n)$ onto the centralizer algebra of the complex symplectic group $\mathrm{Sp(2k, \mathbb{C})}$ acting on the f–fold tensor power of its vector representation. In both cases $\ker(\Phi) = \mathrm{rad}(\tau_n)$, by [18, Corollary 3.5]. Thus Theorem 7.9 also describes the structure of $\Phi(B_f(\mathbb{C}, n))$.

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