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ASYMPTOTIC BEHAVIOR IN A DETERMINISTIC EPIDEMIC MODEL

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The effects of a periodic contact rate and of carriers are considered for a generalization of Bailey's simple epidemic model. In this model it is assumed that individuals become susceptible again as soon as they recover from the infection so that a fixed population can be divided into a class of infectives and a class of susceptibles which vary with time. If the contact rate is periodic, then the number of infectives as time approaches infinity either tends to zero or is asymptotically periodic depending on whether the total population size is less than or greater than a threshold value. The behavior for large time of the number of infectives is determined for three modifications of the model which involve carriers.

1. Introduction. Deterministic models have a long history of use in the description of the spread of an infection. A basic reference is the monograph of Bailey (1957) which contains a description of both stochastic and deterministic epidemic models. A more recent review of mathematical contributions to the description of the spread of epidemics is by Dietz (1967). A widely used model assumes division of a fixed population into three disjoint classes: susceptibles, infectives and individuals who are removed from the susceptible infective interaction by isolation, death or permanent immunity due to previous infection. This model (hereafter referred to as the SIR model) is not appropriate for diseases where infective individuals become susceptible again as soon as they recover. Consequently, we consider the following model which is a special case of a model studied by Kermack and McKendrick (1932) and generalizes the simple epidemic model proposed by Bailey (1950).

Let a fixed population N be divided into a class of infectives $I(t)$ and a class of susceptibles $S(t)$ which vary with time t such that $S(t) + I(t) = N$ and

$$I(0) = I_0 \neq 0. \tag{1}$$

Let the rate of increase of infectives due to new infections be $\beta I(t)S(t)$ where β is a constant called the infection or contact rate. Thus the rate of new infections is proportional to the number of infectives and the number of susceptibles. Assume that infection does not give immunity and let the rate at which infectives recover and become susceptible again be $\gamma I(t)$ where the recovery rate γ is a constant. Thus the differential equation for $I(t)$ is

$$I'(t) = \beta I(t)S(t) - \gamma I(t). \tag{2}$$

The number of susceptibles can always be found from $I(t)$ by using $S(t) = N - I(t)$. The solution and asymptotic behavior of this model (which we call the SIS model) and of the corresponding stochastic model were determined by Weiss and Dishon (1971).

The first modification of the SIS model considered is a periodic contact rate $\beta(t)$. If the population size N is greater than the average relative recovery rate $\bar{\rho}$ (defined below), then $I(t)$ is asymptotic as $t \to \infty$ to a periodic function. If $N \leq \bar{\rho}$, then $I(t)$ tends to zero in an oscillatory manner. The other modifications of the SIS model involve dissemination of the infection by carriers. If the infection is spread either by both infectives and a constant number of carriers or by carriers only, then $I(t)$ is asymptotic as $t \to \infty$ to a positive constant. If the number of carriers decreases as an exponential function of time, then $I(t)$ tends to zero.

These modified models of SIS type might be appropriate for respiratory diseases when infection gives only type specific immunity or when immunity is rapidly lost. Periodic variation in the number of infectives has been observed in illnesses transmitted via the respiratory tract such as common cold, influenza, pneumonia, streptococcal sore throat and meningitis (Rogers, 1963). Carriers are a mode of transmission in meningitis and streptococcal sore throat (Benenson, 1970).

2. *Periodic Contact Rate.* Let the contact or infection rate in the differential equation (2) be a positive continuous periodic function $\beta(t)$ with period p and let the recovery rate γ be constant. For seasonal variation, the period p would be 1 year. Using $S(t) = N - I(t)$ in (2), the differential equation for $I(t)$ is

$$I'(t) = [\beta(t)N - \gamma]I(t) - \beta(t)I^2(t). \tag{3}$$

For linear differential equations with periodic coefficients, the solutions have periodic factors, usually in a product with an exponential. Since the coefficients in the nonlinear differential equation (3) are periodic, one would expect periodic factors in the solution. The periodic and exponential components of $I(t)$ are identified below and then the asymptotic behavior as $t \to \infty$ of $I(t)$ is given. The results presented in this Section and in Section 3 are proved in Section 4.

Let $k = N\bar{\beta} - y$ where $\bar{\beta} = 1/p \int_0^p \beta(u)\, du$. The function

$$\alpha(t) = N \int_0^t [\beta(u) - \bar{\beta}]\, du$$

is easily shown to be periodic with period p. The unique solution for $t \geq 0$ of (1) and (3) is

$$I(t) = \frac{\exp\left[N \int_0^t \beta(u)\, du - \gamma t\right]}{\int_0^t \beta(v) \exp\left[N \int_0^v \beta(u)\, du - \gamma v\right] dv + 1/I_0} \tag{4}$$

$$= \frac{\exp[\alpha(t)]}{\theta(t) + \exp(-kt)/I_0}, \tag{5}$$

where

$$\theta(t) = \int_0^t \exp[-k(t - v)]\beta(v) \exp[\alpha(v)]\, dv. \tag{6}$$

The behavior of $I(t)$ is clarified by considering its asymptotic behavior. For $t \geq p$,

$$\theta(t) = \begin{cases} \varphi(t) - D \exp(-kt)/(\exp(kp) - 1) & k \neq 0 \quad (7) \\ \chi(t) + [t/p]D & k = 0 \quad (8) \end{cases}$$

where

$$\varphi(t) = \frac{D \exp\{-k(t - [t/p]p - p)\}}{\exp(kp) - 1} + \theta(t - [t/p]p) \tag{9}$$

and

$$\chi(t) = \theta(t - [t/p]p). \tag{10}$$

The symbol $[t/p]$ is used to denote the greatest integer that does not exceed t/p. The constant D is defined in Section 4. It can be verified directly that $\varphi(t)$ and $\chi(t)$ are periodic functions with period p.

The behavior of $I(t)$ for large t with $k < 0$, $k = 0$ and $k > 0$ is obtained by using (5)–(10). Complete asymptotic expansions (Erdélyi, 1956) have been obtained, but are not included here. If $k = N\bar{\beta} - \gamma \leq 0$, the infective class $I(t) \to 0$ in an oscillatory manner as $t \to \infty$. For $k = N\bar{\beta} - \gamma > 0$, the asymptotic representation for $I(t)$ is $\exp[\alpha(t)]/\varphi(t)$ which is a periodic function with period p. If we call $\bar{\rho} = \gamma/\bar{\beta}$ the average relative recovery rate, then a principal result of this paper can be stated as follows: $I(t)$ is asymptotically periodic if $N > \bar{\rho}$ and $I(t)$ tends to zero if $N \leq \bar{\rho}$. The average relative recovery rate $\bar{\rho}$ could be called a threshold value since the asymptotic behavior of the epidemic depends on whether the total population N is less than or greater than $\bar{\rho}$. This result is similar to the Kermack and McKendrick (1927, 1932) Threshold Theorems. Graphs which illustrate the asymptotic behavior of the epidemic are given in Figures 1 and 2.

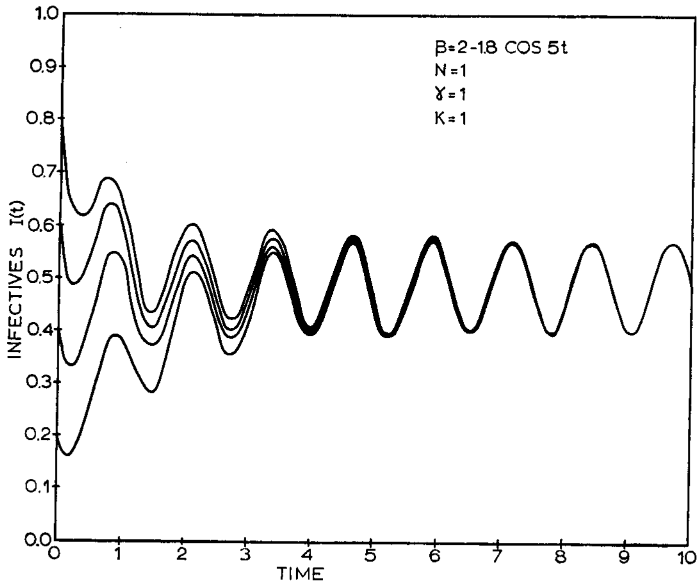

Figure 1. Solution curves of (3) are shown with $k = 1 > 0$ and I_0 = 0,2, 0.4, 0.6 and 0.8. The curves are asymptotically periodic with period $2\pi/5$

If β is a constant, then the above results simplify considerably. The unique solution of (1) and (2) is

$$I(t) = \begin{cases} \dfrac{\exp(kt)}{\beta(e^{kt} - 1)/k + 1/I_0} & k \neq 0 \\ \dfrac{1}{\beta t + 1/I_0} & k = 0. \end{cases} \tag{11}$$

The behavior of $I(t)$ for large t with $k < 0$, $k = 0$ and $k > 0$ is obtained directly from (11). Since $k = N\beta - \gamma$, if we let $\rho = \gamma/\beta$, then $I(t) \to N - \rho$ as $t \to \infty$

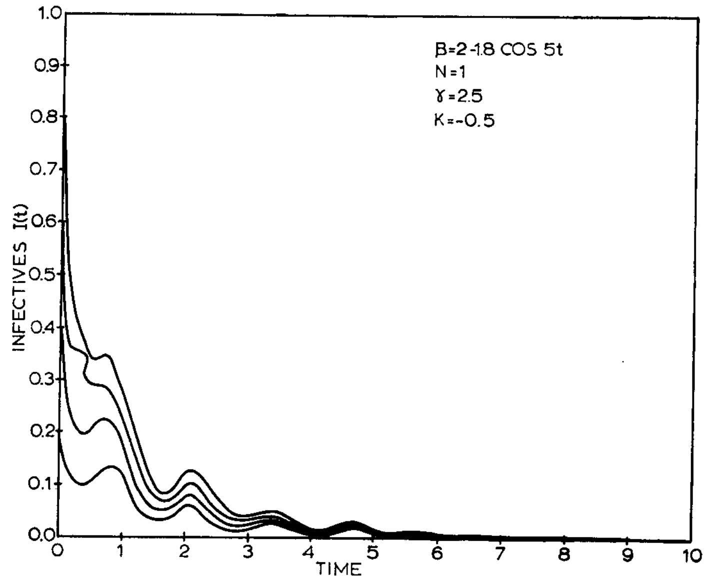

Figure 2. Solution curves of (3) with $k = -0.5 < 0$ and $I_0 = 0.2$, 0.4, 0.6 and 0.8 are shown. The curves approach zero in an oscillatory manner

for $N > \rho$ and $I(t) \to 0$ as $t \to \infty$ for $N \leq \rho$. These results were obtained earlier by Weiss and Dishon (1971). Asymptotic expansions of $I(t)$ have also been obtained, but are not presented here.

If the contact rate β is constant or periodic in the SIR model, $I(t) \to 0$ as $t \to \infty$ for all values of N and ρ, and $S(t)$ approaches a positive constant. Recurrent behavior in an SIR model modified to allow a constant growth rate for the susceptible class and an equal death rate in the removed class has been studied by using perturbations near the equilibrium solution. These and other results on recurrent epidemics are described in Bailey (1957, Chap. 8). The birth of new susceptibles is essential in obtaining a nonzero equilibrium point for the differential equations and periodic behavior in the SIR model. Birth and death within the fixed total population would not affect the results in the SIS model of this section unless a significant portion of deaths were due to the infectious disease. If one assumes that the total population grows exponentially in the SIS model so that $S(t) + I(t) = N \exp(at)$ and that all individuals contributing to the growth are susceptible, then $I(t) \to \infty$ as $t \to \infty$. The same conclusion holds for linear growth of the total population.

3. Carriers. Assume that the infection is spread both by infectives and by a constant number of carriers C. Then the differential equation given by (2) is changed to

$$\begin{aligned} I' &= \beta(I + C)S - \gamma I \\ &= \beta CN + \beta(N - C - \rho)I - \beta I^2 \end{aligned} \tag{12}$$

where $\rho = \gamma/\beta$. As before, the susceptibles can be found by using $S(t) = N - I(t)$. We remark that the rate of new infections $\beta CS(t)$ in (12) could be due to an inanimate carrier such as a polluted water supply which infects at a rate proportional to the number of susceptibles.

The unique solution for $t \geq 0$ of (1) and (12) is

$$I(t) = \frac{1}{\beta}\frac{\alpha_1(\beta I_0 + \alpha_2)e^{\alpha_1 t} + \alpha_2(\beta I_0 - \alpha_1)e^{-\alpha_2 t}}{(\beta I_0 + \alpha_2)e^{\alpha_1 t} - (\beta I_0 - \alpha_1)e^{-\alpha_2 t}} \tag{13}$$

where

$$\alpha_{1,2} = \frac{\beta}{2}\{[(N - C - \rho)^2 + 4CN]^{1/2} \pm (N - C - \rho)\}$$

with α_1 and α_2 corresponding to the plus and minus signs, respectively. From (13) we see that the asymptotic representation of $I(t)$ is α_1/β so that $I(t)$ is asymptotic to a positive constant for all values of N and ρ. This is reasonable since one would expect the carriers to keep the number of infectives from tending to zero. An asymptotic expansion of $I(t)$ has also been obtained.

If the infection in the SIS model is spread only by a constant number of carriers C, then the differential equation (2) becomes $I'(t) = \beta C(N - I) - \gamma I$ which is a first order linear differential equation. The solution is easily found to be

$$I(t) = \left(I_0 - \frac{\beta CN}{\beta C + \gamma}\right)e^{-(\beta C + \gamma)t} + \frac{\beta CN}{\beta C + \gamma}$$

so that $I(t)$ is asymptotic to the positive constant $\beta CN/(\beta C + \gamma)$. In the SIR model, when the infection is spread either by both infectives and a constant number of carriers or by carriers only, both $S(t)$ and $I(t)$ approach zero as t approaches infinity; i.e. the total population becomes removed since the carriers eventually infect everyone.

Weiss (1965) considers the deterministic and stochastic behavior of a model (of SIR type or of SIS type with $\gamma = 0$) in which only carriers spread the disease and the number of carriers decreases exponentially with time as the carriers are identified and eliminated. Using these assumptions, the differential equation (2) becomes

$$\begin{aligned} I'(t) &= \beta(C\,e^{-at})S - \gamma I \\ &= \beta CN\,e^{-at} - (\beta C\,e^{-at} + \gamma)I. \end{aligned}$$

The solution of this linear first order equation and (1) is easily found to be

$$I(t) = \frac{\beta C N \int_0^t \exp(-av - \beta C\, e^{-av}/a + \gamma v)\, dv + I_0\, e^{-\beta C/a}}{\exp(-\beta C\, e^{-at}/a + \gamma t)}.$$

It can then be shown that $I(t) \to 0$ as $t \to \infty$. We would expect the number of infectives to tend to zero as the carriers are eliminated since only carriers spread the infection.

4. Proofs of the Results in Sections 2 and 3. It can be verified directly that (4) and (13) are global solutions of (1) and (3) and (1) and (12). Since the right-hand sides of the differential equations (3) and (12) are locally Lipschitzian, solutions are locally unique by the Picard theorem (Birkhoff and Rota, 1962). Hence (4) and (13) are the unique global solutions of (1) and (3) and (1) and (12). The solutions (4) and (13) were found by using $I = w'/\beta w$ in the Riccati differential equations (3) and (12) and then solving the resulting second order linear differential equations for w.

The equation (5) for $I(t)$ is obtained from (4) by using $\theta(t)$ as defined by (6) and

$$N \int_0^t \beta(u)\, du - \gamma t = N\bar{\beta} t + \alpha(t) - \gamma t = kt + \alpha(t).$$

By dividing the integral $\theta(t)$ into the sum of an integral from 0 to p and an integral from p to t, and then changing variables in the latter integral, one obtains the recursion relation

$$\theta(t) = D \exp(-kt) + \theta(t - p)$$

for $t \geq p$ where

$$D = \int_0^p e^{kv} \beta(v)\, e^{\alpha(v)}\, dv.$$

For $k \neq 0$, we use the recursion relation to find

$$\theta(t) = D\, e^{-kt}(1 + e^{kp} + \cdots + e^{kp[t/p]}) + \theta(t - [t/p]p)$$

$$= D\, e^{-kt} \frac{\exp\{kp([t/p] + 1)\} - 1}{e^{kp} - 1} + \theta(t - [t/p]p).$$

This yields (7) with $\varphi(t)$ defined by (9). For $k = 0$, a similar method yields (8) with $\chi(t)$ defined by (10). If β is constant, then (11) follows from (4).

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