2.8: Approximating solution using

**Method of Successive Approximation**

(also called Picard's iteration method).

IVP: $y' = f(t, y)$, $y(t_0) = y_0$.

Note: Can always translate IVP to move initial value to the origin and translate back after solving:

Hence for simplicity **in section 2.8**, we will assume initial value is at the origin: $y' = f(t, y)$, $y(0) = 0$.

**Thm 2.4.2:** Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Thm 2.8.1 is translated to origin version of Thm 2.4.2:

**Thm 2.8.1:** Suppose the functions
$z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous for all $t$ in $(-a, a) \times (-c, c)$,
then there exists an interval $(-h, h) \subset (-a, a)$ such that there exists a unique function $y = \phi(t)$ defined on $(-h, h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(0) = 0.$$

**Proof outline** (note this is a constructive proof and thus the proof is very useful).

Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)
$f$, $\partial f/\partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$.

Then $y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0$ iff

$\int_0^t \phi'(s) ds = \int_0^t f(s, \phi(s)) ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s)) ds$

Thus $y = \phi(t)$ is a solution to (*)
iff $\phi(t) = \int_0^t f(s, \phi(s)) ds$

Construct $\phi$ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

To finish the proof, need to answer the following questions (see book or more advanced class):

1.) Does $\phi_n(t)$ exist for all $n$?

2.) Does sequence $\phi_n$ converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y.$ That is $f(t, y) = t + 2y$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t s ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

Let $\phi_4(t) = \int_0^t f(s, \phi_3(s))ds$

$$= \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6})ds$$

$$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6}))ds$$

$$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$$

$\vdots$

Determine formula for $\phi_n$:

Note patterns:

$\int_0^t s ds = \frac{t^2}{2} =$

$\int_0^t \frac{s^2}{2} ds = \frac{t^3}{3 \cdot 2} =$

$\int_0^t \frac{s^3}{3 \cdot 2} ds = \frac{t^4}{4 \cdot 3 \cdot 2} =$

$\int_0^t \frac{s^4}{4 \cdot 3 \cdot 2} ds = \frac{t^5}{5 \cdot 4 \cdot 3 \cdot 2} =$

Thus look for factorials.

$\phi_0(t) = 0$

$\phi_1(t) = \frac{t^2}{2}$

$\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

$\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15} = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{3 \cdot 2} + \frac{t^5}{5 \cdot 3}$

Thus $\phi_n(t) =$

FYI (ie not on quizzes/exam):

Defn: $\sum_{k=0}^{\infty} a_k x^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ....$$

Taylor's Theorem: If $f$ is analytic at 0, then for small $x$ (i.e., $x$ near 0),

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \tfrac{f''(0)}{2} x^2 + \tfrac{f'''(0)}{6} x^3 + ...$$

Example:

$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$ and thus $e^{bt} = \sum_{k=0}^{\infty} \frac{b^k t^k}{k!}$ for $t$ near 0.

$$\phi_n(t) = \sum_{k=2}^{n} \frac{2^{k-2}}{k!} t^k$$

Thus $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=2}^{\infty} \frac{2^{k-2}}{k!} t^k = \frac{1}{4} \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k$

$$= \frac{1}{4} ( \qquad - \qquad - \qquad )$$