7.6: Complex eigenvalue example: Solve $\mathbf{x}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \mathbf{x}$

Step 1 Find eigenvalues: $det(A - rI) = 0$

$$det(A - rI) = \begin{vmatrix} 3-r & -13 \\ 5 & 1-r \end{vmatrix} = (3-r)(1-r) + 65 = r^2 - 4r + 68 = 0$$

Thus $r = \frac{4 \pm \sqrt{4^2 - 4(68)}}{2} = \frac{4 \pm \sqrt{4(4-68)}}{2} = \frac{4 \pm 2\sqrt{-64}}{2} = 2 \pm 8i$

Step 2 Find eigenvectors: Solve $(A - rI)\mathbf{x} = \mathbf{0}$

$$A - (2 \pm 8i)I = \begin{bmatrix} 3 - (2 \pm 8i) & -13 \\ 5 & 1 - (2 \pm 8i) \end{bmatrix} = \begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix}$$

Solve $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix} = \begin{bmatrix} (1 \mp 8i)13 - 13(1 \mp 8i) \\ 5(13) + (-1 \mp 8i)(1 \mp 8i) \end{bmatrix} = \begin{bmatrix} 0 \\ 65 + (-1 + 64i^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus any non-zero multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ is an eigenvector of A with eigen value $2 \pm 8i$.

Note: $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ since $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus we can use either $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ or any nonzero multiple.

General solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} cos(8t) - \begin{bmatrix} 8 \\ 0 \end{bmatrix} sin(8t) \right) + c_2 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} sin(8t) + \begin{bmatrix} 8 \\ 0 \end{bmatrix} cos(8t) \right)$$

Slope field for x_2 vs x_1: $\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{x_2'}{x_1'} = \frac{5x_1 + x_2}{3x_1 - 13x_2}$

Note slope 0's occur when $5x_1 + x_2 = 0$, ie, $x_2 = -5x_1$.

Note slope ∞'s occur when $3x_1 - 13x_2 = 0$, ie, $x_2 = \frac{3}{13}x_1$.

Determine where slopes are positive vs negative for regions between these lines:

For example, along the positive x_1 axis slope is positive: $x_2 = 0$ and $\frac{dx_2}{dx_1} = \frac{5x_1}{3x_1} = \frac{+}{+}$, $x_1 > 0$

For example, along the positive x_2 axis slope is negative: $x_1 = 0$ and $\frac{dx_2}{dx_1} = \frac{x_2}{-13x_2} = \frac{+}{-}$, $x_2 > 0$