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Ch 2.2: Separable Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order equation

$$\frac{dy}{dx} = f(x, y)$$

- We can rewrite this in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

- For example, let $M(x,y) = -f(x,y)$ and $N(x,y) = 1$. There may be other ways as well. In differential form,

$$M(x, y)dx + N(x, y)dy = 0$$

- If M is a function of x only and N is a function of y only, then

$$M(x)dx + N(y)dy = 0$$

- In this case, the equation is called **separable**.

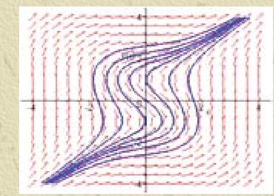
Example 1: Solving a Separable Equation

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2+1}{y^2-1}$$

- Separating variables, and using calculus, we obtain

$$(y^2-1)dy = (x^2+1)dx$$
$$\int (y^2-1)dy = \int (x^2+1)dx$$
$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + C$$
$$y^3 - 3y = x^3 + 3x + C$$



- The equation above defines the solution y implicitly. A graph showing the direction field and implicit plots of several integral curves for the differential equation is given above.

Example 2: Implicit and Explicit Solutions (1 of 4)

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$

- Separating variables and using calculus, we obtain

$$2(y-1)dy = (3x^2+4x+2)dx$$
$$2\int (y-1)dy = \int (3x^2+4x+2)dx$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

- The equation above defines the solution y implicitly. An explicit expression for the solution can be found in this case:

$$y^2 - 2y - (x^3+2x^2+2x+C) = 0 \;\Rightarrow\; y = \frac{2 \pm \sqrt{4+4(x^3+2x^2+2x+C)}}{2}$$
$$y = 1 \pm \sqrt{x^3+2x^2+2x+C}$$

Example 2: Initial Value Problem (2 of 4)

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain

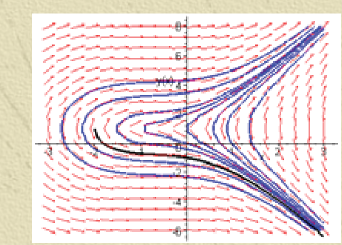
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$
$$(-1)^2 - 2(-1) = C \;\Rightarrow\; C = 3$$

- Thus the implicit equation defining y is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

- Using explicit expression of y,

$$y = 1 \pm \sqrt{x^3+2x^2+2x+C}$$
$$-1 = 1 \pm \sqrt{C} \;\Rightarrow\; C = 4$$



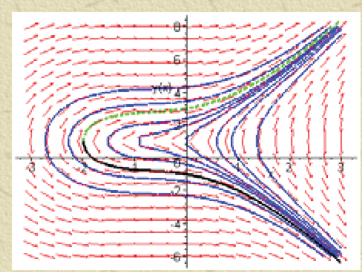
- It follows that

$$y = 1 - \sqrt{x^3+2x^2+2x+4}$$

Example 2: Initial Condition $y(0) = 3$ (3 of 4)

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$



Example 2: Domain (4 of 4)

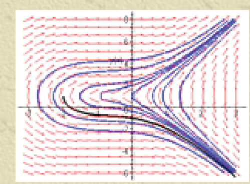
- Thus the solutions to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

are given by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \quad \text{(implicit)}$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{(explicit)}$$



- From explicit representation of y, it follows that

$$y = 1 - \sqrt{x^2(x+2) + 2(x+2)} = 1 - \sqrt{(x+2)(x^2+2)}$$

and hence domain of y is $(-2, \infty)$. Note $x = -2$ yields $y = 1$, which makes denominator of dy/dx zero (vertical tangent).
- Conversely, domain of y can be estimated by locating vertical tangents on graph (useful for implicitly defined solutions).

Example 3: Implicit Solution of Initial Value Problem (1 of 2)

- Consider the following initial value problem:

$$y' = \frac{y \cos x}{1 + 3y^3}, \quad y(0) = 1$$

- Separating variables and using calculus, we obtain

$$\frac{1 + 3y^3}{y} dy = \cos x dx$$

$$\int \left(\frac{1}{y} + 3y^2 \right) dy = \int \cos x dx$$

$$\ln|y| + y^3 = \sin x + C$$

- Using the initial condition, it follows that

$$\ln y + y^3 = \sin x + 1$$

Example 3: Graph of Solutions (2 of 2)

- Thus

$$y' = \frac{y \cos x}{1 + 3y^3}, \quad y(0) = 1 \Rightarrow \ln y + y^3 = \sin x + 1$$

- The graph of this solution (black), along with the graphs of the direction field and several integral curves (blue) for this differential equation, is given below.