Math 3600 Differential Equations Exam #1
March 2, 2016 SHOW ALL WORK

[12] 1.) Find the largest eigenvalue and its corresponding eigenvectors for $\begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix}$

$\begin{vmatrix} 5-r & 2 \\ 3 & 0-r \end{vmatrix} = (5-r)(-r) - 6 = r^2 - 5r - 6 = (r+1)(r-6)$. Thus $r = -1, 6$

For $r = 6$: $\begin{bmatrix} 5-r & 2 \\ 3 & 0-r \end{bmatrix} = \begin{bmatrix} 5-6 & 2 \\ 3 & 0-6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Answer: The largest eigenvalue of the above matrix is 6

and its eigenvectors are all non-zero multiples of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

[8] 2.) Find all the singular points of the following differential equation and determine whether each one is regular or irregular.

$$x^3(x-3)y'' - 6xy' + 9xy = 0$$

$1y'' - \frac{6}{x^2(x-3)}y' + \frac{9}{x^2(x-3)}y = 0$. Thus $x = 0, 3$ are singular points.

Euler equation: $x^2y'' + \alpha xy' + \beta y = 0$.

Multiply by x^2: $x^2y'' - \left(\frac{6}{x(x-3)}\right)xy' + \frac{9}{(x-3)}y = 0$. Thus $x = 0$ is an irregular singular point.

Multiply by $(x-3)^2$: $(x-3)^2y'' - \left(\frac{6}{x^2}\right)(x-3)y' + \frac{9(x-3)}{x^2}y = 0$. Thus $x = 3$ is an regular singular point.

Alternately: $lim_{x\to 0}\ x(\frac{6x}{x^2(x-3)})$ is not finite. Thus $x = 0$ is an irregular singular point.

$lim_{x\to 3}\ (x-3)(\frac{6}{x^2(x-3)})$ and $lim_{x\to 3}\ (x-3)^2(\frac{9}{x^2(x-3)})$ are finite. Thus $x = 3$ is an regular singular point.

[20] 3.) Solve

$$y'' - 6y' + 9y = \frac{e^{3t}}{t}$$

Solve homogeneous equation: $y'' - 6y' + 9y = 0$.

Guess $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$

$r^2 - 6r + 9 = (r-3)^2 = 0$. Thus $r = 3$

Thus general homogeneous solution is $y = c_1 e^{3t} + c_2 te^{3t}$

Solve non-homogeneous equation: $y'' - 6y' + 9y = \frac{e^{3t}}{t}$

General non-homogeneous solution is $y = c_1 e^{3t} + c_2 te^{3t} + (u_1(t)e^{3t} + u_2(t)(te^{3t}))$ where we determine u_i as follows:

$$W(e^{3t}, te^{3t}) = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t} + 3te^{3t} \end{vmatrix} = e^{6t} + 3te^{6t} - 3te^{6t} = e^{6t}$$

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & te^{3t} \\ 1 & e^{3t} + 3te^{3t} \end{vmatrix}}{e^{6t}} (\tfrac{e^{3t}}{t})dt = \int \tfrac{-te^{3t}}{e^{6t}} (\tfrac{e^{3t}}{t})dt = \int \tfrac{-te^{6t}}{te^{6t}} dt = -\int dt = -t + C_1$$

$$u_2(t) = \int \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & 1 \end{vmatrix}}{e^{6t}} (\tfrac{e^{3t}}{t})dt = \int \tfrac{e^{3t}}{e^{6t}} (\tfrac{e^{3t}}{t})dt = \int \tfrac{dt}{t} = ln|t| + C_2$$

NOTE: need only 1 solution for each u_i and thus don't need constants C_1, C_2.

Alternate method to find u_i: $y = u_1 e^{3t} + u_2 te^{3t}$

Then $y' = u_1' e^{3t} + 3u_1 e^{3t} + u_2' te^{3t} + u_2[e^{3t} + 3te^{3t}]$

Choose 2nd equation for solving for 2 unknowns u_1, u_2: Let $u_1' e^{3t} + +u_2' te^{3t} = 0$

Then $y' = 3u_1 e^{3t} + u_2[e^{3t} + 3te^{3t}]$

and $y'' = 3u_1' e^{3t} + 9u_1 e^{3t} + u_2'[e^{3t} + 3te^{3t}] + u_2[3e^{3t} + 3e^{3t} + 9te^{3t}]$

$y'' = 3u_1' e^{3t} + u_2'[e^{3t} + 3te^{3t}] + 9u_1 e^{3t} + u_2[6e^{3t} + 9te^{3t}]$

$y'' = (3u_1' e^{3t} + u_2'[3te^{3t}]) + u_2'[e^{3t}] + 9u_1 e^{3t} + u_2[6e^{3t} + 9te^{3t}]$

$y'' = 3(0) + u_2' e^{3t} + 9u_1 e^{3t} + u_2[6e^{3t} + 9te^{3t}] = u_2' e^{3t} + 9u_1 e^{3t} + u_2(6e^{3t} + 9te^{3t})$

Plug into DE:

$u_2' e^{3t} + 9u_1 e^{3t} + u_2(6e^{3t} + 9te^{3t}) - 6[3u_1 e^{3t} + u_2(e^{3t} + 3te^{3t})] + 9[u_1 e^{3t} + u_2 te^{3t}] = \frac{e^{3t}}{t}$

$u_2' e^{3t} + (9u_1 e^{3t} - 18u_1 e^{3t} + 9u_1 e^{3t}) + [6u_2 e^{3t} + 9u_2 te^{3t} - 6u_2 e^{3t} - 18u_2 te^{3t} + 9u_2 te^{3t}] = 0$

$u_2' e^{3t} = \frac{e^{3t}}{t}$. Thus $u_2' = \frac{1}{t} = t^{-1}$ and $u_2(t) = ln|t|$

$u_1' e^{3t} + u_2' te^{3t} = 0$ implies $u_1' e^{3t} + \frac{1}{t} te^{3t} = 0$

Thus $u_1' = -1$ and $u_1(t) = -t$

$y = c_1 e^{3t} + c_2 te^{3t} + (-te^{3t} + ln|t|(te^{3t})) = c_1 e^{3t} + (c_2 - 1)te^{3t} + ln|t|(te^{3t})$

Answer: $\underline{y = c_1 e^{3t} + c_2 te^{3t} + ln|t|(te^{3t})}$

[20] 4.) Solve $x^2y'' + 8xy' - 8y = 0.$

Guess $y = x^r$. Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$

Plug $y = x^r$ into $x^2y'' + 8xy' - 8y = 0.$

$x^2r(r-1)x^{r-2} + 8xrx^{r-1} - 8x^r = 0.$

$(r^2 - r)x^r + 8rx^r - 8x^r = 0.$

$(r^2 - r + 8r - 8)x^r = 0.$

$(r^2 + 7r - 8)x^r = 0$ implies $(r-1)(r+8) = 0$. Thus $r = 1, -8$

Thus $y = x$ and $y = x^{-8}$ are two linearly independent solutions to this 2nd order linear homogeneous DE.

Answer: $\underline{\; y = c_1x + c_2x^{-8} \;}$

[20] 5.) Seek the power series solution for the given first order differential equation about the point $x_0 = 0$:

$$(x+4)y' - 2y = 0$$

Note: A solution for this problem is a finite polynomial and thus all but a finite number of terms will be 0, but you must plug in the infinite series.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

$$(x+4)y' - 2y = (x+4)\sum_{n=0}^{\infty} a_n n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n$$

$$= x\sum_{n=0}^{\infty} a_n n x^{n-1} + 4\sum_{n=0}^{\infty} a_n n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 4a_n n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=1}^{\infty} 4a_n n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 4a_{n+1}(n+1) x^n - \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=0}^{\infty} (a_n n + 4a_{n+1}(n+1) - 2a_n) x^n$$

$$= \sum_{n=0}^{\infty} (a_n(n-2) + 4a_{n+1}(n+1)) x^n = 0$$

Thus $a_n(n-2) + 4a_{n+1}(n+1) = 0$ for all $n \geq 0$

$a_{n+1} = \frac{2-n}{4(n+1)} a_n$

$n = 0$: $a_1 = \frac{2}{4} a_0 = \frac{1}{2} a_0$

$n = 1$: $a_2 = \frac{2-1}{4(2)} a_1 = \frac{1}{8} a_1 = \frac{1}{8}(\frac{1}{2} a_0) = \frac{1}{16} a_0$

$n = 2$: $a_3 = \frac{2-2}{4(3)} a_2 = 0$

$a_{n+1} = \frac{2-n}{4(n+1)}(0) = 0$ for $n > 2$

Thus $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = a_0 + \frac{1}{2} a_0 x + \frac{1}{16} a_0 x^2 = a_0(1 + \frac{1}{2}x + \frac{1}{16}x^2)$

Check: $y' = a_0(\frac{1}{2} + \frac{1}{8}x)$

$(x+4)y' - 2y = (x+4)(a_0)(\frac{1}{2} + \frac{1}{8}x)) - 2a_0(1 + \frac{1}{2}x + \frac{1}{16}x^2)$

$= a_0(\frac{1}{2}x + \frac{1}{8}x^2 + 2 + \frac{1}{2}x - 2 - x - \frac{1}{8}x^2) = 0$

Answer: $\underline{y = a_0(1 + \frac{1}{2}x + \frac{1}{16}x^2)}$

[20] 6.) Given the recursive relation $a_{n+2} = 6a_{n+1} - 9a_n$ where $a_0 = -1$ and $a_1 = 3$, prove that $a_n = 3^n(2n-1)$. You may use induction.

Proof by induction: First we prove that $a_n = 3^n(2n-1)$ for $n = 0, 1$:

$n = 0$: $3^0(2(0) - 1) = -1 = a_0$

$n = 1$: $3^1(2(1) - 1) = 3 = a_1$

Induction hypothesis: Suppose $a_k = 3^k(2k-1)$ for $k = n, n+1$.

Then $a_n = 3^n(2n-1)$ and $a_{n+1} = 3^{n+1}(2(n+1)-1)$

Claim: $a_{n+2} = 3^{n+2}(2(n+2)-1)$

$$\begin{aligned}
a_{n+2} &= 6a_{n+1} - 9a_n \\
&= 6[3^{n+1}(2(n+1)-1)] - 9[3^n(2n-1)] \\
&= 2[3^{n+2}(2n+2-1)] - 3^{n+2}(2n-1) \\
&= 2[3^{n+2}(2n+1)] - 3^{n+2}(2n-1) \\
&= 3^{n+2}(4n+2) - 3^{n+2}(2n-1) \\
&= 3^{n+2}[4n+2-2n+1] \\
&= 3^{n+2}[2n+3] \\
&= 3^{n+2}[2(n+2)-1]
\end{aligned}$$

Alternative answer (**not** covered in this class – see MATH:4050 Intro to Discrete Math):

Guess $a_n = x^n$. Then $a_{n+1} = x^{n+1}$ and $a_{n+2} = x^{n+2}$

Then $a_{n+2} = 6a_{n+1} - 9a_n$ implies $x^{n+2} - 6x^{n+1} + 9x^n = 0$.

Hence $x^n(x^2 - 6x + 9) = x^n(x-3)^2 = 0$. Thus $x = 3$

Claim: $a_n = c_1(3^n) + c_2(n3^n)$ satisfies $a_{n+2} - 6a_{n+1} + 9a_n = 0$

$$\begin{aligned}
&c_1(3^{n+2}) + c_2((n+2)3^{n+2}) - 6[c_1(3^{n+1}) + c_2((n+1)3^{n+1})] + 9[c_1(3^n) + c_2(n3^n)] \\
&= c_1[3^{n+2} - 6(3^{n+1}) + 9(3^n)] + c_2[(n+2)3^{n+2} - 6((n+1)3^{n+1}) + 9(n3^n)] \\
&= c_1[3^{n+2} - 2(3^{n+2}) + (3^{n+2})] + c_2\{n[(3^{n+2} - 6(3^{n+1}) + 9(3^n)] + [(2)3^{n+2} - 6(3^{n+1})]\} \\
&= c_1[0] + c_2\{n[(3^{n+2} - 2(3^{n+2}) + (3^{n+2})] + [(2)3^{n+2} - 2(3^{n+2})]\} = 0
\end{aligned}$$

Thus $a_n = c_1(3^n) + c_2(n3^n)$

IVP: $a_0 = -1$ and $a_1 = 3$,

$n = 0$: $-1 = c_1(3^0) + c_2(0) = c_1$

$n = 1$: $3 = c_1(3) + c_2(3) = -3 + 3c_2$. Thus $c_2 = 2$

Hence $a_n = -(3^n) + 2(n3^n) = 3^n(2n - 1)$