Approximating soln to y' = t + 2y, y(0) = 0 using slope field.

But don't get algebra eqn just get a picture

Slope fields can be misleading especially when multiple soln or no soln or computer errors

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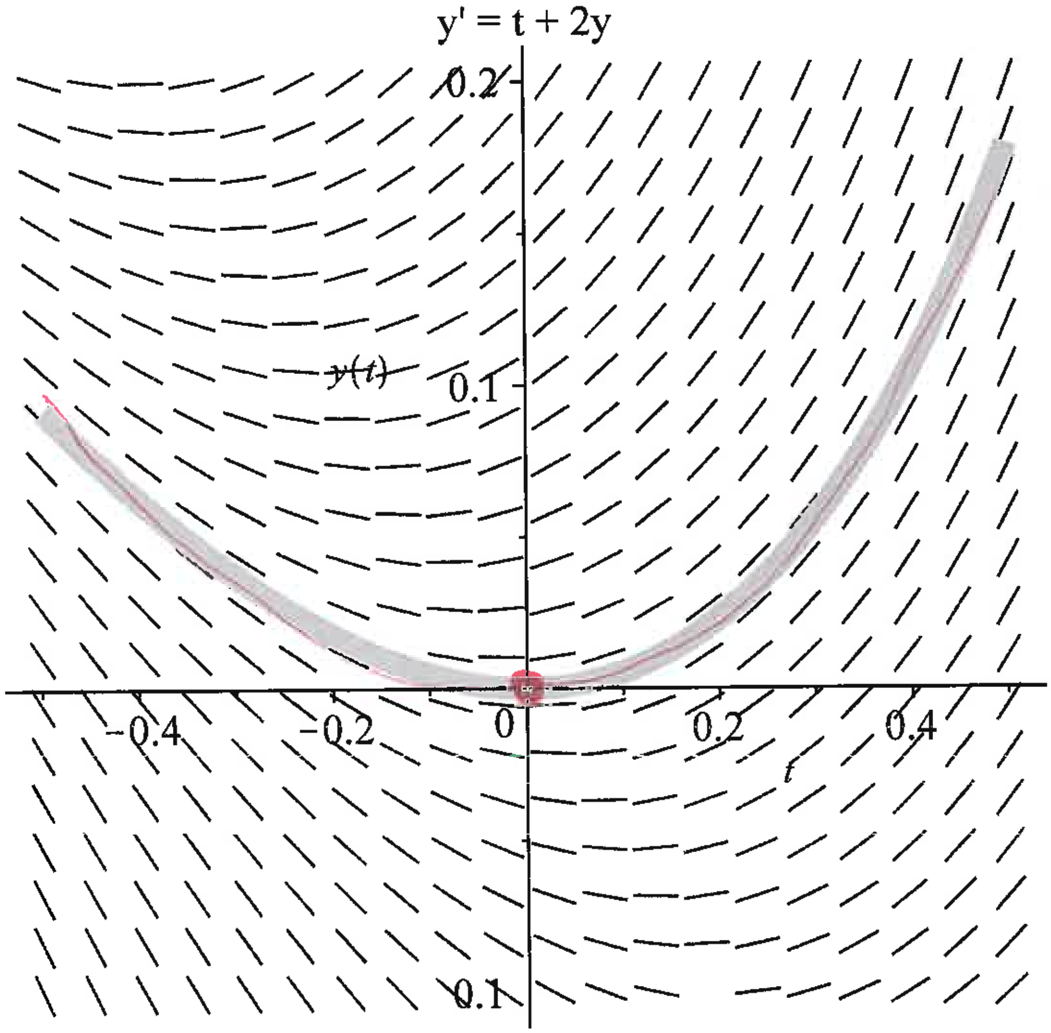
> with( DEtools, odeadvisor ) :
> with( plots ) :
> ode1 := diff( y(t), t ) = t + 2*y(t);
```

$$ode1 := \frac{d}{dt}\, y(t) = t + 2\, y(t) \quad (1)$$

```
> DEplot(ode1, [y(t)], t=-0.5..0.5, y=-0.1..0.2, arrows=LINE, color=purple, title
      ="y' = t + 2y", {[0, 0]}, thickness=9, linecolor=cyan );
```



Approximating soln to y' = t + 2y, y(0) = 0 using Picard's iteration method.

$\int y' = \int f(t,y)$

$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$

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> odeadvisor(ode1, y(t))
```

$$odeadvisor\left(\frac{d}{dt}\,y(t) = t + 2\,y(t), y(t)\right) \quad (2)$$

```
> dsolve(ode1, y(t));
```

$$y(t) = -\frac{t}{2} - \frac{1}{4} + e^{2t}\_C1 \quad (3)$$

```
> ans := rhs(dsolve({ode1, y(0) = 0}));
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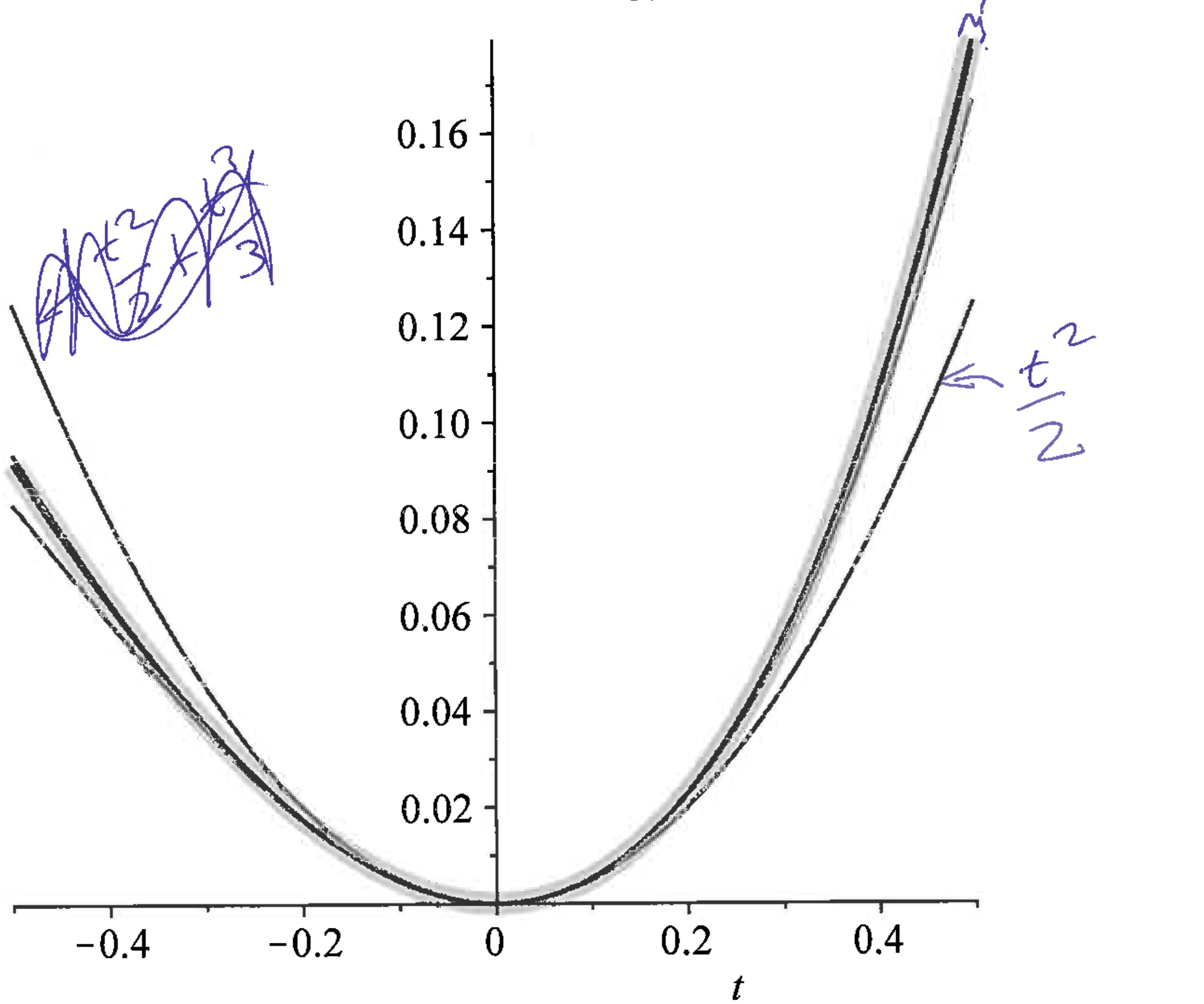
$$ans := -\frac{t}{2} - \frac{1}{4} + \frac{e^{2t}}{4} \quad (4)$$

```
> plots[multiple](plot, [ans, t=-0.5..0.5, thickness=9, color=cyan], [t^2/2, t=-0.5..0.5, color=red], [t^2/2 + t^3/3, t=-0.5..0.5, color=brown], [t^2/2 + t^3/3 + t^4/6, t=-0.5..0.5, color=blue], [t^2/2 + t^3/3 + t^4/6 + t^5/15, t=-0.5..0.5, color=black])
```



$\int_0^t y' = \int_0^t f(s, y)\,ds$

$$\boxed{\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds}$$

2.8: Approximating solution using **Method of Successive Approximation** (also called Picard's iteration method).

IVP: $y' = f(t, y)$, $y(t_0) = y_0$.

Note: Can always translate IVP to move initial value to the origin and translate back after solving:

Hence for simplicity **in section 2.8**, we will assume initial value is at the origin: $y' = f(t, y)$, $y(0) = 0$.

Thm 2.4.2: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Thm 2.8.1 is translated to origin version of Thm 2.4.2:

Thm 2.8.1: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous for all t in $(-a, a) \times (-c, c)$,
then there exists an interval $(-h, h) \subset (-a, a)$ such that there exists a unique function $y = \phi(t)$ defined on $(-h, h)$ that satisfies the following initial value problem: $\boxed{y' = f(t, y); \quad y(0) = 0.}$

Proof outline (note this is a constructive proof and thus the proof is very useful).

Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)
f, $\partial f / \partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$.

Then $y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t))$, $\phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s))ds$, $\phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s))ds$

Thus $y = \phi(t)$ is a solution to (*)
iff $\phi(t) = \int_0^t f(s, \phi(s))ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

To finish the proof, need to answer the following questions (see book or more advanced class):

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Let $\phi_0(t) = 0$, $\phi_{n+1} = \int_0^t f(s, \phi_n(s))ds$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

Let $\phi_4(t) = \int_0^t f(s, \phi_3(s))ds$

$= \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6})ds$

$= \int_0^t (s + 2(\frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{6}))ds$

$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$

$\vdots$

$0, \frac{t^2}{2}, \frac{t^2}{2} + \frac{t^3}{3}, \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} \xrightarrow{\text{as } n\to\infty} \phi$

$\phi_0, \phi_1, \phi_2, \ldots$ approx approach soln

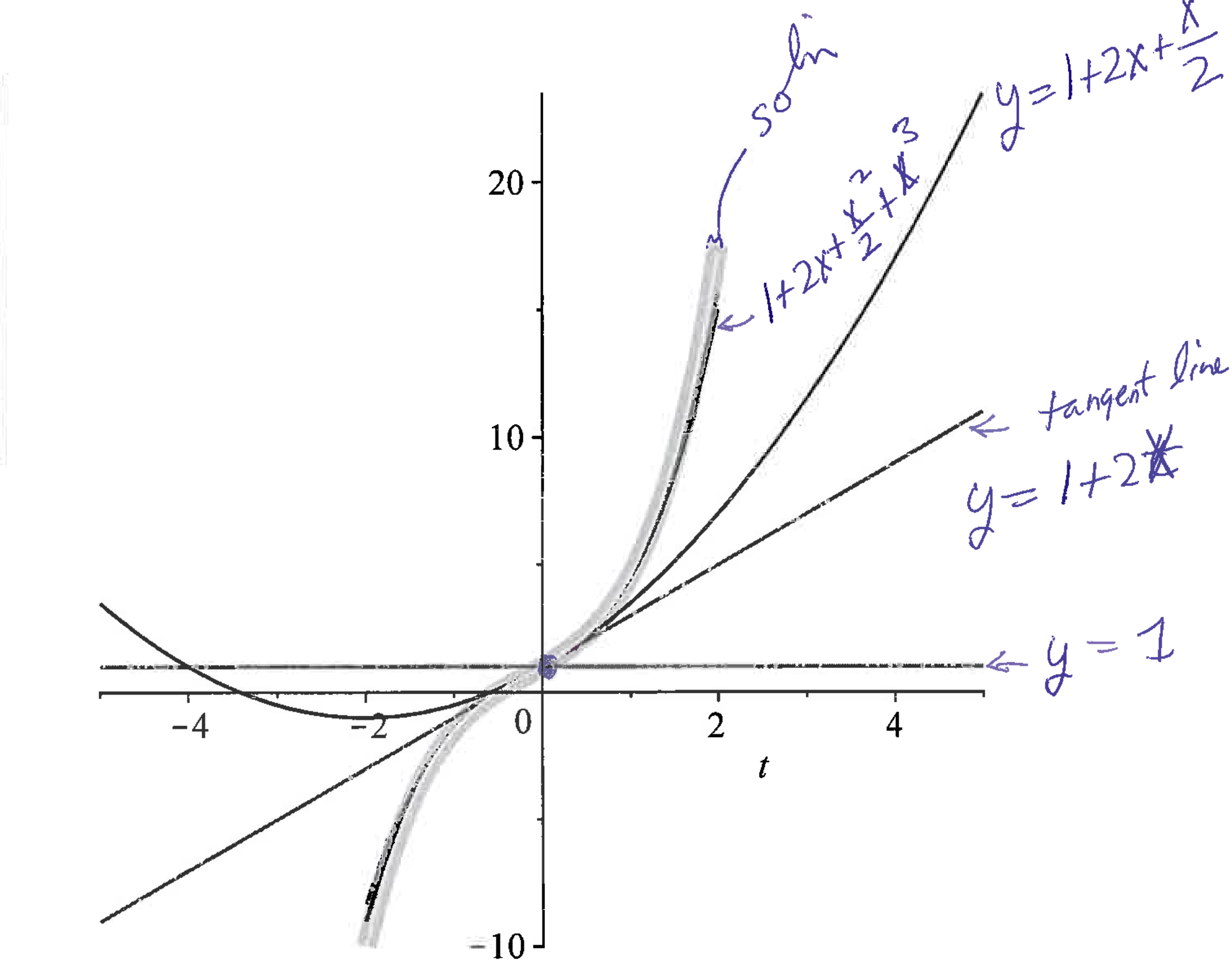
$y = \sum a_n t^n$ if ordinary pt at $t=0$

Approximating soln to $y'' - y = 4t$, $y(0) = 1$, $y'(0) = 2$ using series approximation (ch 5).

> $ans := -4 \cdot t - \frac{5\exp(-t)}{2} + \frac{7\exp(t)}{2}$

$$ans := -4\,t - \frac{5\,e^{-t}}{2} + \frac{7\,e^{t}}{2} \tag{5}$$

> $plots[multiple]\Big(plot, [ans, t=-2..2, thickness=9, color=cyan], [1, t=-5..5, color=red], [1 + 2\,t, t=-5..5, color=brown], \Big[1 + 2\,t + \frac{t^2}{2}, t=-5..5, color=blue\Big], \Big[1 + 2\,t + \frac{t^2}{2} + t^3, t=-2..2, color=black\Big], \Big[1 + 2\,t + \frac{t^2}{2} + t^3 + \frac{t^4}{24}, t=-2..2, color=orange\Big], \Big[1 + 2\,t + \frac{t^2}{2} + t^3 + \frac{t^4}{24} + \frac{6\,t^5}{120}, t=-2..2, color=pink\Big]\Big)$



[20] 4.) Using power series, find a degree 5 polynomial approximation for the solution to $y''-y=4x$ for x near 0

$y=\Sigma_{n=0}^{\infty}a_nx^n$, $y'=\Sigma_{n=1}^{\infty}a_nnx^{n-1}$, $y''=\Sigma_{n=2}^{\infty}a_nn(n-1)x^{n-2}$

$x=0$ ~~at~~ ordinary pt

$\Sigma_{n=2}^{\infty}a_nn(n-1)x^{n-2}-\Sigma_{n=0}^{\infty}a_nx^n=4x$

$\Sigma_{n=2}^{\infty}a_nn(n-1)x^{n-2}-\Sigma_{n=0}^{\infty}a_nx^n=4x$

plug in $y=\sum_{n=0} a_n x^n$

$\Sigma_{n=0}^{\infty}a_{n+2}(n+2)(n+1)x^n-\Sigma_{n=0}^{\infty}a_nx^n=4x$

$\Sigma_{n=0}^{\infty}[a_{n+2}(n+2)(n+1)-a_n]x^n=4x$

2nd order $\Rightarrow$ 2 constants

For $n=1$: $[a_3(3)(2)-a_1]x=4x$. Thus $a_3=\frac{a_1+4}{6}$

$\boxed{a_0, a_1}$

For $n\neq 1$, $a_{n+2}(n+2)(n+1)-a_n=0$. Thus $a_{n+2}=\frac{a_n}{(n+2)(n+1)}$

two constants

For $n=0$: $a_2=\frac{a_0}{(2)(1)}$

For $n=2$: $a_4=\frac{a_2}{(4)(3)}=\frac{a_0}{(4)(3)(2)(1)}$

Find $a_2, a_3, \ldots$

For $n=3$: $a_5=\frac{a_3}{(5)(4)}=\frac{a_1+4}{6(5)(4)}$

Approximation: $\boxed{y=a_0+a_1x+\frac{a_0}{2}x^2+\frac{a_1+4}{6}x^3+\frac{a_0}{4!}x^4+\frac{a_1+4}{120}x^5}$ $\leftarrow \sum_{n=0}^{5} a_n x^n$

[22] 5.) Solve $y''-y=e^t+2$, $y(0)=1$, $y'(0)=2$

Solve homogeneous: Guess $y=e^{rt}$ and plug into $y''-y=0$: $r^2e^{rt}-e^{rt}=0$.

Thus $r^2-1=(r+1)(r-1)=0$. Thus $r=1,-1$.

Homogeneous solution: $c_1e^t+c_2e^{-t}$

Solve $y''-y=e^t$

$y=e^t$ is a homogeneous solution, so guess $y=Ate^t$. Then $y'=Ae^t+Ate^t$ and $y''=Ae^t+Ae^t+Ate^t=2Ae^t+Ate^t$.

Plug into $y''-y=e^t$:

$2Ae^t+Ate^t-Ate^t=e^t$ implies $2Ae^t=e^t$. Thus $2A=1$ and $A=\frac{1}{2}$.

Thus $y=\frac{1}{2}te^t$ is one solution to $y''-y=e^t$

Solve $y''-y=2$

Guess $y=B$, then $y'=0, y''=0$.

Plug in: $0-B=2$. Thus $B=-2$.

Thus $y=-2$ is one solution to $y''-y=2$

Hence general solution to $y''-y=e^t+2$ is $y=c_1e^t+c_2e^{-t}+\frac{1}{2}te^t-2$

Solve IVP: $y(0)=1$, $y'(0)=2$

$y=c_1e^t+c_2e^{-t}+\frac{1}{2}te^t-2$