Week 9		
10/19	[Abel, 7.3, 7.1](#) [(pdf)](#), [annotated](#) [(pdf)](#)	HW 8 (due Sunday 10/25) 4.1: 1, 4, 5, 7, 16 4.2: 4, 13, 16 4.3: 2, 7, 10 - 13 7.3: 13, 14, 15 7.1 (use matrix form): 3, 4, 5, 6, 12 Quiz 3 (due Sunday 10/25) [3 points, unlimited attempts]
10/21	ch 3, 4, 7 defns, 7.1, 7.2	
10/23	Review	
Week 10		
10/26	Problem session	Tentative HW 9 (due Friday 10/30) and ??
10/28	Exam 2	

Abel's theorem: if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

then $W(\phi_1, \phi_2, ..., \phi_n)(t) = ce^{-\int p_1(t)dt}$ for some constant c

Ex: Find the Wronskian of a fundamental set of solutions of the DE

$y'' + 5y' = 0$

Method 1: Find homogeneous solution

$r^2 + 5r = 0$ implies $r = 0, -5$

homog sol'n $y = c_1 e^{0t} + c_2 e^{-5t} = c_1(1) + c_2 e^{-5t} = c_1 + c_2 e^{-5t}$

A fundamental set of solutions: $\{1, e^{-5t}\}$

Wronskian $= W(1, e^{-5t})(t) = det \begin{pmatrix} 1 & e^{-5t} \\ 0 & -5e^{-5t} \end{pmatrix} = -5e^{-5t}$

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

$y'' + 5y' = 0$ implies $p_1(t) = 5$.

Thus Wronskian $= W(1, e^{-5t})(t) = ce^{-\int 5dt} = ce^{-5t}$

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

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$y'' + 5y' = 0$ implies $p_1(t) = 5$.

Thus Wronskian $= W(1, e^{-5t})(t) = ce^{-\int 5dt} = ce^{-5t}$

Chapter 7: Systems of Linear DE

Defn: A set V together with two operations, called addition and scalar multiplication is a **vector space** if the following vector space axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars, c, d in R.

Vector space axioms:

a.) $\mathbf{u} + \mathbf{v}$ is in V

b.) $c\mathbf{u}$ is in V

c.) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

d.) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

e.) There is a vector, denoted by $\mathbf{0}$, in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V

f.) For each $\mathbf{u}$ in V, there is an element, denoted by $-\mathbf{u}$, in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

g.) $(cd)\mathbf{u} = c(d\mathbf{u})$

h.) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

i.) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

j.) $1\mathbf{u} = \mathbf{u}$

Linear Algebra Review: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the linear transformation $T : V \to V$ if there exists a nonzero vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\mathbf{x}$.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is <u>NOT</u> an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\text{Thus } A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A\left(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $\lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(\lambda I - A)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(\lambda I - A) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(\lambda_0 I - A)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(\lambda I - A) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$.

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

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Chapter 7: Systems of Linear DE

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = 4x_1 + x_2,$$
$$x_2'(t) = 5x_1$$

Write in matrix form:

Homogeneous vs non-homogeneous system of linear DE:

7.1: Transforming an n^{th} order linear DE into a system of n first order linear DEs.

Ex: $y'' - 5y' + 6y = 0$

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Ex: $y'' - 5y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 2$

7.1: Transforming an n^{th} order linear DE into a system of n first order linear DEs.

Ex: $y'' - 5y' + 6y = \sin(t)$

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