Is $f(x) = 2^x$ linear?

$f(0+0) = \cancel{2^0}\ f(0) = 2^0 = 1$
$f(0)+f(0) = 2^0 + 2^0 = 1+1 = 2$
$\Big\}\ 1 \neq 2 \Rightarrow f$ is not linear

Linear Functions

A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if 1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f: R \to R$, $f(x) = 2x$

Proof:
$f(ax+by) = 2(ax+by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f: R \to R$, $f(x) = 2x+3$ is NOT linear.

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0+1) = f(1) = 5$, but $f(0)+f(1) = 3+5 = 8$. Hence $f(0+1) \neq f(0)+f(1)$

Note confusing notation: Most lines, $f(x) = mx+b$ are not linear functions.

Question: When is a line, $f(x) = mx+b$, a linear function?

Example 2.) $f: R^2 \to R^2$,
$f((x_1, x_2)) = (2x_1, x_1 + x_2)$

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x}+b\mathbf{y} = a(x_1,x_2)+b(y_1,y_2) = (ax_1, ax_2)+(by_1,by_2) = (ax_1+by_1, ax_2+by_2)$

$$\begin{aligned} f(ax_1+by_1, ax_2+by_2) &= (2(ax_1+by_1), ax_1+by_1+ax_2+by_2) \\ &= (2ax_1+2by_1, ax_1+ax_2+by_1+by_2) \\ &= (2ax_1, ax_1+ax_2) + (2by_1, by_1+by_2) \\ &= a(2x_1, x_1+x_2) + b(2y_1, y_1+y_2) \\ &= af((x_1,x_2)) + bf((y_1,y_2)) \end{aligned}$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af+bg) = (af+bg)' = af'+bg' = aD(f)+bD(g)$

$D(2x^{3}\cancel{3} + \sin x) = D(2x^3) + D(\sin x)$
$= 2D(x^3) + D(\sin x) = 2 \cdot 3x^2 + \cos x$
$= 6x^2 + \cos x$

$af'' + bf' + cf = 0$ homogen

ψ_1 is a soln $\Leftrightarrow L(\psi_1) = 0$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to R$,
$I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

s, t scalars, f, g are functions

Proof:

$$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$$
$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$
$$= \underline{s}(\underline{af'' + bf' + cf}) + \underline{t}(\underline{ag'' + bg' + cg})$$
$$= sL(f) + tL(g)$$

Consequence 1: If ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ψ_1, ψ_2 are solutions to $af''+bf'+cf = 0$, $L(\psi_1) = 0$ and $L(\psi_2) = 0$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$
$= 3(0) + 5(0) = 0.$

Thus $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

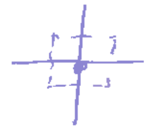
Since ψ_1 is a solution to $af''+bf'+cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af''+bf'+cf = k$, $L(\psi_2) = k$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$
$= 3h + 5k.$

Thus $3\psi_1 + 5\psi_2$ is also a solution to
$af'' + bf' + cf = 3h + 5k$

2.8



Given: $\boxed{y' = f(t,y),\ y(0) = 0}$ Eqn (*)

$f, \partial f/\partial y$ continuous $\forall (t,y) \in (-a,a) \times (-b,b)$. Then

$y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s,\phi(s)ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s,\phi(s)ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\boxed{\phi(t) = \int_0^t f(s,\phi(s)ds}$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s,\phi_0(s)ds$

Let $\phi_2(t) = \int_0^t f(s,\phi_1(s)ds$

⋮

Let $\boxed{\phi_{n+1}(t) = \int_0^t f(s,\phi_n(s)ds}$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

For specific case:

Some questions:

week 15

1.) Does $\phi_n(t)$ exist for all n? Pf by induction

2.) Does sequence ϕ_n converge? Example: Ratio test

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*). Plug it in

4.) Is the solution unique. ← week 15

Example: $y' = t + 2y$. That is $f(t,y) = t + 2y$

$\frac{\partial f}{\partial y} = 2$ cont $\forall (t,y)$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s,0)ds = \int_0^t (s + 2(0))ds$

$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$

Let $\phi_2(t) = \int_0^t f(s,\phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$

Let $\phi_3(t) = \int_0^t f(s,\phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

⋮

See class notes.

$\phi_n(t) = \sum_{i=1}^{n} \frac{2^{i-1} t^{i+1}}{(i+1)!}$