

Recall  $\phi_1, \dots, \phi_n$  are linearly independent iff  $c_1 = \dots = c_n = 0$  is the only solution to  $c_1\phi_1(t) + \dots + c_n\phi_n(t) = 0$ .

If  $\phi_i$  are functions of  $t$ , then  $\mathbf{0}$  is the constant function,  $\mathbf{0}(t) = 0$  for all  $t$ . Thus  $c_1\phi_1(t) + \dots + c_n\phi_n(t) = 0$  for all  $t$ .

Hence  $c_1\phi_1^{(k)}(t) + \dots + c_n\phi_n^{(k)}(t) = 0$  for all  $t$ ,  $k$  if derivatives exist.

Thus  $\phi_1, \dots, \phi_n$  are linearly independent iff for any given  $f$ ,  $c_1\phi_1(t) + \dots + c_n\phi_n(t) = 0$  has a unique solution (that works for all  $t$ ).

iff the following system of equations has a unique solution

$$\begin{aligned} c_1\phi_1(t) + c_2\phi_2(t) + \dots + c_n\phi_n(t) &= 0 \\ c_1\phi_1'(t) + c_2\phi_2'(t) + \dots + c_n\phi_n'(t) &= 0 \\ \vdots & \\ c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + \dots + c_n\phi_n^{(n-1)}(t) &= 0 \end{aligned}$$

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \dots & \phi_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some  $t_0$

$$\det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff  $W(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$ ,

Example: Determine if  $\{1+2t, 5+4t^2, 6-8t+8t^2\}$  are linearly independent:

No, not lin indep

$$\begin{vmatrix} 1 & 5 & 6 \\ 0 & -10 & -20 \\ 0 & 4 & 8 \end{vmatrix} = 0$$

Method 1: Solve  $c_1(1+2t) + c_2(5+4t^2) + c_3(6-8t+8t^2) = 0 = 0t^2 + 0t + 0$

Or equivalently, solve  $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or equivalently, solve  $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Method 2: Check the Wronskian

$n$ th order LINEAR differential equation:

Theorem 4.1.1: If  $p_i : (a, b) \rightarrow R$ ,  $i = 1, \dots, n$  and  $g : (a, b) \rightarrow R$  are continuous and  $a < t_0 < b$ , then there exists a unique function  $y = \phi(t)$ ,  $\phi : (a, b) \rightarrow R$  that satisfies the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof: We proved the case  $n = 1$  using an integrating factor. When  $n > 1$ , see more advanced textbook.

Claim: If  $p_i$  are continuous on  $(a, b)$ , if  $\phi_1, \dots, \phi_n$  are linearly independent solutions to  $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0$ , then  $\{\phi_1, \dots, \phi_n\}$  is a basis for the solution set to this differential equation.

Theorem 4.1.2: If  $p_i$  are continuous on  $(a, b)$ , suppose that  $\phi_i, i = 1, \dots, n$  are solutions to  $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0$ . If  $W(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$ , for some  $t_0 \in (a, b)$ , then any solution to this homogeneous linear differential equation can be written as

$$y = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$$

Defn: The  $\phi_1, \dots, \phi_n$  are called a fundamental set of solutions to  $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0$ .

Theorem: Given any  $n$ th order homogeneous linear differential equation, there exist a set of  $n$  functions which form a fundamental set of solutions.

### 4.1: General Theory of nth Order Linear Equations

*Thm:*  $L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y$  is a linear function.

*Proof:* Let  $a, b$  be real numbers.

$$\begin{aligned} & \boxed{L(af + bg)} \\ &= (af + bg)^{(n)} + p_1(t)(af + bg)^{(n-1)} + \dots + p_{n-1}(t)(af + bg)' + p_n(t)(af + bg) \\ &= af^{(n)} + bg^{(n)} + p_1(t)(af^{(n-1)} + bg^{(n-1)}) + \dots + p_{n-1}(t)(af' + bg') + p_n(t)(af + bg) \\ &= af^{(n)} + p_1af^{(n-1)} + \dots + p_{n-1}af' + p_naf + bg^{(n)} + p_1bg^{(n-1)} + \dots + p_{n-1}bg' + p_nbg \\ &= a[f^{(n)} + p_1f^{(n-1)} + \dots + p_{n-1}f' + p_nf] + b[g^{(n)} + p_1g^{(n-1)} + \dots + p_{n-1}g' + p_ng] \\ &= \boxed{aL(f) + bL(g)} \end{aligned}$$

*Theorem:* If  $\phi_i, i = 1, \dots, n$  are solutions to a homogeneous linear differential equation (i.e.,  $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0$  (\*)), then  $c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$  is also a solution to this linear differential equation.

*Pf:* Since  $\phi_i, i = 1, \dots, n$  are solutions to (\*),  $L(\phi_i) = 0$  for  $i = 1, \dots, n$ . Thus  $L(c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n) = c_1L(\phi_1) + c_2L(\phi_2) + \dots + c_nL(\phi_n) = 0$ . Thus  $c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$  is also a solution to (\*).

Solve:  $y'' + y = 0, y(0) = -1, y'(0) = -3$

$r^2 + 1 = 0$  implies  $r^2 = -1$ . Thus  $r = \pm i$ .

Since  $r = 0 \pm 1i, y = k_1\cos(t) + k_2\sin(t)$ . Then  $y' = -k_1\sin(t) + k_2\cos(t)$

$y(0) = -1: -1 = k_1\cos(0) + k_2\sin(0)$  implies  $-1 = k_1$

$y'(0) = -3: -3 = -k_1\sin(0) + k_2\cos(0)$  implies  $-3 = k_2$

Thus IVP solution:  $y = -\cos(t) - 3\sin(t)$

**When does the following IVP have a unique solution:**

IVP:  $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0,$

$y(t_0) = y_0, y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}$ .

Suppose  $y = c_1\phi_1(t) + c_2\phi_2(t) + \dots + c_n\phi_n(t)$  is a solution to this IVP. Then

$y(t_0) = y_0: y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \dots + c_n\phi_n(t_0)$

$y'(t_0) = y_1: y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + \dots + c_n\phi_n'(t_0)$

$y^{(n-1)}(t_0) = y_{n-1}: y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + \dots + c_n\phi_n^{(n-1)}(t_0)$

To find IVP solution, need to solve above system of equations for the unknowns  $c_i, i = 1, \dots, n$ .

Note the IVP has a unique solution if and only if the above system of equations has a unique solution for the  $c_i$ .

Note that in these equations the  $c_i$  are the unknowns and the  $y_i, \phi_i(t_0), \dots, \phi_i^{(n-1)}(t_0)$ , are the constants.

We can translate this linear system of equations into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$\det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

*Definition:* The Wronskian of the functions,  $\phi_1, \phi_2, \dots, \phi_n$  is

$$W(\phi_1, \phi_2, \dots, \phi_n) = \det \begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \dots & \phi_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{bmatrix}$$

*Theorem:* Suppose that  $\phi_i, i = 1, \dots, n$  are solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0.$$

If  $W(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$ , then there is a unique choice of constants  $c_i$  such that  $c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$  satisfies this homogeneous linear differential equation and initial conditions,  $y(t_0) = y_0, y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}$ .

normal solutions and two will be the normal solution each multiplied by  $t$ .

From: Paul's online notes

The general solution is,

$$y(t) = c_1 + c_2 e^{-3t} \cos(5t) + c_3 e^{-3t} \sin(5t) + c_4 t e^{-3t} \cos(5t) + c_5 t e^{-3t} \sin(5t)$$

Let's now work an example that contains all three of the basic cases just to say that we've got one work here.

**Example 4** Solve the following differential equation. Guess  $y = e^{rt}$

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

**Solution**

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2(r^2 - 4r + 5) = 0$$

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

In this case we've got one real distinct root,  $r = 1$ , and double root,  $r = 5$ , and a pair of complex roots,  $r = 2 \pm i$  that only occur once.

The general solution is then,

$$y(t) = c_1 e^t + c_2 e^{5t} + c_3 t e^{5t} + c_4 e^{2t} \cos(t) + c_5 e^{2t} \sin(t)$$

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_1 \\ y''(t_0) &= y_2 \\ y'''(t_0) &= y_3 \\ y^{(4)}(t_0) &= y_4 \end{aligned}$$

We've got one final example to work here that on the surface at least seems almost too easy. The problem here will be finding the roots as well see.

**Example 5** Solve the following differential equation.

$$y^{(4)} + 16y = 0$$

**Solution**

The characteristic equation is

$$r^4 + 16 = 0$$

~~$y(t) = c_1 e^{rt} + c_2 e^{st} + c_3 e^{it} + c_4 e^{-it}$~~