5.5 Series Solutions Near a Regular Singular Point, Part I

Theorem 5.3.1: If $p(x)$ and $q(x)$ are analytic at x_0 (i.e., x_0 is an ordinary point of the ODE $y''+p(x)y'+q(x)y=0$), then the general solution to this ODE is

$$\boxed{y = \Sigma_{n=0}^{\infty} a_n(x-x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)}$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

$p = \frac{Q}{P}$, $\frac{R}{P} = q$

~~If~~ you prefer a power series expansion about 0, use u-substitution: let $u = x - x_0$. Then $p(u+x_0)$ and $q(u+x_0)$ are analytic at 0

$y = \Sigma a_n x^n$

(Semi-failed) attempt to transform 5.5 problem into 5.4 problem:

5.5: $y'' + p(x)y' + q(x)y = 0$

$x^2y'' + x^2p(x)y' + x^2q(x)y = 0$

$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$ where $xp(x)$ and $x^2q(x)$ are functions of x.

$\boxed{\textbf{5.4: } x^2y'' + \alpha xy' + \beta y = 0 \textbf{ where } \alpha, \beta \textbf{ are constants.}}$ ← guessed $|x|^r$

Combine 5.3/5.4 methods.

Defn: x_0 is a *regular singular value* if x_0 is a singular value and $xp(x)$ and $x^2q(x)$ are analytic at x_0. A singular value which is not regular is called *irregular*.

bad but not too bad

Examples:

$y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$, regular singular value: $x = 0$.

$y'' + \frac{y'}{x^2} + \frac{y}{x} = 0$, irregular singular value: $x = 0$.

$y'' + y' + \frac{y}{x^3} = 0$, irregular singular value: $x = 0$.

$x\left(\frac{1}{x^2}\right)$ & $x^2\left(\frac{1}{x}\right)$

bad ✓

Is $\lim_{x\to 0} xp(x)$ finite & is $\lim_{x\to 0} x^2q(x)$ finite?

yes ⇒ regular singular value ; No ⇒ irreg sing value

If $p(x)$ and $q(x)$ are rational functions, then $xp(x)$ and $x^2q(x)$ are analytic iff $lim_{x\to 0}xp(x)$ and iff $lim_{x\to 0}x^2q(x)$ are finite. (i.e., after reducing fractions, x is not in the denominator.

Ex: $p(x) = \frac{1}{x}$ implies $xp(x) = \frac{x}{x} = 1$ ← ok

Ex: $p(x) = \frac{1}{x^2}$ implies $xp(x) = \frac{x}{x^2} = \frac{1}{x}$ ← $\lim_{x\to 0} \frac{1}{x} = \infty$

If $x_0 = 0$ is a regular singular value of the linear homogeneous DE, $x^2y'' + x[xp(x)]y' + x^2q(x)y = 0$ (*), then $xp(x) = \Sigma_{n=0}^{\infty} p_n x^n$ and $x^2q(x) = \Sigma_{n=0}^{\infty} q_n x^n$ for constants p_n, q_n.

for regular sing pt

If $y = x^r\Sigma_{n=0}^{\infty} a_n x^n = \Sigma_{n=0}^{\infty} a_n x^{n+r}$ is a solution to (*) where $r \neq 0$.

$y' = \Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$ and $y'' = \Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$

$$x^2\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + x[xp(x)]\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + [x^2q(x)]\Sigma_{n=0}^{\infty}a_nx^{n+r}$$

$$\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + [xp(x)]\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r} + [x^2q(x)]\Sigma_{n=0}^{\infty}a_nx^{n+r}$$

$$\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + (\Sigma_{n=0}^{\infty}p_nx^n)(\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r}) + (\Sigma_{n=0}^{\infty}q_nx^n)(\Sigma_{n=0}^{\infty}a_nx^{n+r})$$

Thus the coefficient of x^r is $r(r-1)a_0 + p_0ra_0 + q_0a_0 = 0$

a_0x^{n+r} is first nonzero term in analytic expansion

We can take $a_0 \neq 0$. Thus $r(r-1) + p_0r + q_0 = 0$

Thus we can solve for r using the quadratic formula.

Case 1: $r_1 > r_2$ both real and $r_1 - r_2$ is not an integer.
Case 2: $r_1 > r_2$ both real and $r_1 - r_2 = p$, p an integer.
Case 3: one repeated root.
Case 4: two complex roots.

$xp(x)$ & $x^2q(x)$ are analyt

5.5: Solve $x^2y'' - x(2+x)y' + (2+x^2)y = 0$ $\;\rightarrow\;$ $y'' - \underbrace{\left(\frac{2+x}{x}\right)}_{p(x)} y' + \underbrace{\left(\frac{2+x^2}{x^2}\right)}_{q(x)} y = 0$

$p(x) = -\frac{x(2+x)}{x^2} = -\frac{2+x}{x}$. Thus $x_0 = 0$ is a singular value. $\quad x\,p(x) = \sum_{n=0}^{\infty} p_n x^n = -(2+x)$

$q(x) = \frac{2+x^2}{x^2}$ also implies $x_0 = 0$ is a singular value. $\quad 2 + x^2 = x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$

$xp(x) = -(2+x)$ and $x^2q(x) = 2 + x^2$. Thus $x_0 = 0$ is a regular singular value.

But our infinite sums are finite in this case

Suppose $\boxed{y = \sum_{n=0}^{\infty} a_n x^{n+r}}$ is a solution. WLOG assume $a_0 \neq 0$ (otherwise one can reindex the summation).

Then $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

$x^2y'' - x(2+x)y' + (2+x^2)y$

$= x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}\right) - (2x+x^2)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right) + (2+x^2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)$

$= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}$
$\quad + \sum_{n=0}^{\infty} 2a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2}$

$= \Sigma_{n=0}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} - \Sigma_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \Sigma_{n=2}^{\infty}a_{n-2}x^{n+r}$

$n=0 \qquad n=1 \qquad n=1$

$= [r(r-1) - 2r + 2]a_0x^r + [(1+r)r - 2(1+r) + 2]a_1x^{r+1} - ra_0x^{r+1}$
$\quad + \Sigma_{n=2}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} - \Sigma_{n=2}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \Sigma_{n=2}^{\infty}a_{n-2}x^{n+r}$

$= [r(r-1) - 2r + 2]a_0x^r + \big([(1+r)r - 2(1+r) + 2]a_1 - ra_0\big)x^{r+1}$
$\quad + \Sigma_{n=2}^{\infty}\big([(n+r)(n+r-1) - 2(n+r) + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\big)x^{n+r}$

Some simplificat

factor out $(n+r)$

$= [r^2 - r - 2r + 2]a_0x^r + \big([r + r^2 - 2 - 2r + 2]a_1 - ra_0\big)x^{r+1}$
$\quad + \Sigma_{n=2}^{\infty}\big([(n+r)(n+r-3) + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\big)x^{n+r}$

factor out x^n

more simplicat

more simplicat

$= [r^2 - 3r + 2]a_0x^r + \big([r^2 - r]a_1 - ra_0\big)x^{r+1}$
$\quad + \Sigma_{n=2}^{\infty}\big([n^2 + 2rn + r^2 - 3n - 3r + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\big)x^{n+r} = 0$

$= \Sigma\, 0\, x^n$

$$\frac{(r^2-3r+2)a_0}{a_0} = \frac{0}{a_0}$$

Set all coefficients $= 0$:

Since $a_0 \neq 0$, $r^2 - 3r + 2 = (r-2)(r-1) = 0$ implies $\boxed{r = 1, 2.}$

$\boxed{r^2 - 3r + 2 = 0 \text{ is the } \textit{indicial equation}}$

$r \neq 0$ since $x = 0$ is a singular point

$[r^2 - r]a_1 = ra_0$ implies $(r-1)a_1 = a_0$. Thus if $r = 1$, $a_0 = 0$, a contradiction. $\boxed{\text{If } r = 2,\ a_1 = a_0}$

$$[n^2+2rn+r^2-3n-3r+2]a_n - (n+r-1)a_{n-1} + a_{n-2} = [n^2+2rn-3n]a_n - (n+r-1)a_{n-1} + a_{n-2} = 0$$

$$a_n = \frac{(n+r-1)a_{n-1}-a_{n-2}}{n^2+2rn-3n} = \frac{(n+1)a_{n-1}-a_{n-2}}{n^2+4n-3n} = \frac{(n+1)a_{n-1}-a_{n-2}}{n^2+n} = \frac{(n+1)a_{n-1}-a_{n-2}}{n(n+1)}$$

$$a_2 = \frac{3a_1-a_0}{6} = \frac{3a_0-a_0}{6} = \frac{2a_0}{6} = \frac{a_0}{3}$$

$$a_3 = \frac{4a_2-a_1}{(3)(4)} = \frac{4(\frac{a_0}{3})-a_0}{(3)(4)} = \frac{4a_0-3a_0}{(3)^2(4)} = \frac{a_0}{(3)^2(4)}$$

$$a_4 = \frac{5a_3-a_2}{(4)(5)} = \frac{(5\frac{a_0}{(3)^2(4)})-(\frac{a_0}{3})}{(4)(5)} = \frac{5a_0-3(4)a_0}{3^2(4)^2(5)} = \frac{7a_0}{3^2(4)^2(5)}$$

$$a_5 = \frac{6a_4-a_3}{(5)(6)} = \frac{6(\frac{7a_0}{3^2(4)^2(5)})-(\frac{a_0}{(3)^2(4)})}{(5)(6)} = \frac{6(7a_0)-(20a_0)}{(3)^2(4)^2(5)^2(6)} = \frac{22a_0}{(3)^2(4)^2(5)^2(6)}$$

$$a_6 = \frac{7a_5-a_4}{(6)(7)} = \frac{7(\frac{22a_0}{(3)^2(4)^2(5)^2(6)})-\frac{7a_0}{3^2(4)^2(5)}}{(6)(7)} = \frac{7(22a_0)-30(7a_0)}{(3)^2(4)^2(5)^2(6)^2(7)} = \frac{-56a_0}{(3)^2(4)^2(5)^2(6)^2(7)}$$

$$a_7 = \frac{8a_6-a_5}{(7)(8)} = \frac{8(\frac{-56a_0}{(3)^2(4)^2(5)^2(6)^2(7)})-\frac{22a_0}{(3)^2(4)^2(5)^2(6)}}{(7)(8)} = \frac{8(-56a_0)-42*22a_0}{(3)^2(4)^2(5)^2(6)^2(7)^2(8)} = \frac{-1372a_0}{(3)^2(4)^2(5)^2(6)^2(7)^2(8)}$$

$$y = x^2(a_0+a_0x+\frac{a_0}{3}x^2+\frac{a_0}{(3)^2(4)}x^3+\frac{7a_0}{3^2(4)^2(5)}x^4+\frac{22a_0}{(3)^2(4)^2(5)^2(6)}x^5+\frac{-56a_0}{(3)^2(4)^2(5)^2(6)^2(7)}x^6+\frac{-1372a_0}{(3)^2(4)^2(5)^2(6)^2(7)^2(8)}x^7+\ldots) \blacksquare$$