In general, to determine if there is a unique solution to the IVP, $y'' - 4y' + 4y = 0$, $y(x_0) = y_0$, $y'(x_0) = y_1$, we solve for unknowns a_0 and a_1.

$$y(x_0) = a_0\phi_0(x_0) + a_1\phi_1(x_0)$$
$$y'(x_0) = a_0\phi_0'(x_0) + a_1\phi_1'(x_0)$$

Note that the above system of two equations has a unique solution for the two unknowns a_0 and a_1 if and only if $det\begin{pmatrix}\phi_0(x_0) & \phi_1(x_0)\\ \phi_0'(x_0) & \phi_1'(x_0)\end{pmatrix} \neq 0$

In other words the IVP has a unique solution iff the Wronskian of ϕ_0 and ϕ_1 evaluated at x_0 is not zero. Recall that by theorem , this also implies that ϕ_0 and ϕ_1 are linearly independent and hence the general solution is $y = a_0\phi_0(x) + a_1\phi_1(x)$ by theorem.

Show that $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent by calculating the Wronskian of these two functions evaluated at $x_0 = 0$.

$$W(\phi_1,\phi_2)(x) = \begin{pmatrix}\phi_1(x) & \phi_2(x)\\ \phi_1'(x) & \phi_2'(x)\end{pmatrix} = \begin{pmatrix}(-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n & \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n\\ (-2)\Sigma_{n=1}^{\infty}\frac{2^{n-1}(n-1)}{(n-1)!x^{n-1}} & \Sigma_{n=1}^{\infty}\frac{n2^{n-1}}{(n-1)!}x^{n-1}\end{pmatrix}$$

$$W(\phi_1,\phi_2)(0) = \begin{pmatrix}(-2)2^{0-1}(-1) & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1 \neq 0$$

Hence $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent

---

When possible identify the functions giving the series solutions. Recall that by Taylor's theorem and the ratio test, $e^{2x} = \Sigma_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}x^n = \Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$ for all x.

$$f(x) = a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$$

$$= a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}}{n!}x^n$$

$$= (a_1 - 2a_0)\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)x\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^{n-1} + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)x\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)xe^{2x} + a_0e^{2x}$$

Note we have recovered the solution we found using the 3.4 method.

Note a power series solutions exists in a neighborhood of x_0 when the solution is analytic at x_0. I.e, the solution is of the form $y = \Sigma_{n=0}^{\infty}a_n(x-x_0)^n$ where this series has a nonzero radius of convergence about x_0.

When do we know an analytic solution exists? I.e, when is this method guaranteed to work?

Special case: $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $y''(x) = -\frac{Q}{P}y' - \frac{R}{P}y$ $\quad y'' + \frac{Q}{P}y' + \frac{R}{P}y$

Definition: The point x_0 is an *ordinary point* of the ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0.

Theorem 5.3.1: If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution to this ODE is

$$y = \Sigma_{n=1}^{\infty}a_n(x-x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

Theorem: If P and Q are polynomial functions, then $y = Q(x)/P(x)$ is analytic at x_0 if and only if $P(x_0) \neq 0$. Moreover if Q/P is reduced, the radius of convergence of $Q(x)/P(x) = min\{||x_0 - x|| \mid x \in \mathbf{C}, P(x) = 0\}$ where $||x_0 - x||$ = distance from x_0 to x in the complex plane.

If x_0 is an ordinary pt.
Translate eqn ~~so~~ so x_0 is at the origin
Then $x_0 = 0$ is an ordinary
Guess $y = \sum_{n=0}^{\infty} a_n x^n$

If t_0 is not an ordinary pt
$\Rightarrow$ singular pt

Series Solutions Near a Regular Singular Point

MATH 365 *Ordinary Differential Equations*

J. Robert Buchanan

Department of Mathematics

Spring 2015

http://banach.millersville.edu/~bob/math365/Singular/main.pdf

Background

We will find a power series solution to the equation:

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

We will assume that t_0 is a **regular singular point**. This implies:

1. $P(t_0) = 0$,
2. $\displaystyle\lim_{t\to t_0} \frac{(t-t_0)Q(t)}{P(t)}$ exists,
3. $\displaystyle\lim_{t\to t_0} \frac{(t-t_0)^2R(t)}{P(t)}$ exists.

If regular singular point, can use ch 5 method
If not regular singular pt
$\Rightarrow$ irregular
$\Rightarrow$ out of luck in this (not covered in class)

Simplification

If $t_0 \neq 0$ then we can make the change of variable $x = t - t_0$ and the ODE:

$$P(x+t_0)y'' + Q(x+t_0)y' + R(x+t_0)y = 0.$$

has a regular singular point at $x = 0$.

From now on we will work with the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

having $\boxed{\text{a regular singular point at } x = 0.}$

Assumptions (1 of 2)

Since the ODE has a regular singular point at $x = 0$ we can define

$$x\frac{Q(x)}{P(x)} = xp(x) \quad \text{and} \quad x^2\frac{R(x)}{P(x)} = x^2q(x)$$

which are analytic at $x = 0$ and

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} xp(x) = p_0$$

$$\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} x^2q(x) = q_0.$$

Practical
(ie you may need to do this)

Assumptions (2 of 2)

Furthermore since $xp(x)$ and $x^2q(x)$ are analytic,

$$xp(x) = \sum_{n=0}^{\infty} p_nx^n$$

$$x^2q(x) = \sum_{n=0}^{\infty} q_nx^n$$

for all $-\rho < x < \rho$ with $\rho > 0$.

Motivation

Re-writing the ODE

The second order linear homogeneous ODE can be written as

$$\begin{aligned}
0 &= P(x)y'' + Q(x)y' + R(x)y \\
&= y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y \\
&= x^2y'' + x^2\frac{Q(x)}{P(x)}y' + x^2\frac{R(x)}{P(x)}y \\
&= x^2y'' + x[xp(x)]y' + [x^2q(x)]y \\
&= x^2y'' + x[p_0 + p_1x + \cdots + p_nx^n + \cdots]y' \\
&\quad + [q_0 + q_1x + \cdots + q_nx^n + \cdots]y.
\end{aligned}$$

$\frac{Q}{P} = p$

$\frac{R}{P} = q$

$y'' + py' + qy$

$x^2y'' + x[xp(x)]y' + [x^2q(x)]y$

Special Case: Euler's Equation

$0 = x^2y'' + x\left[\sum_0^\infty p_n x^n\right]y' + \left[\sum_0^\infty q_n x^n\right]y$

If $p_n = 0$ and $q_n = 0$ for $n \geq 1$ then

$$\begin{aligned} 0 &= x^2y'' + x\left[p_0 + \cancel{p_1x + \cdots + p_nx^n + \cdots}\right]y' \\ &\quad + \left[q_0 + \cancel{q_1x + \cdots + q_nx^n + \cdots}\right]y \\ &= x^2y'' + p_0xy' + q_0y \end{aligned}$$

which is Euler's equation.

$0 = x^2y'' + p_0xy' + q_0y$

Guess: $y = x^r$

$x^2(r)(r-1)x^{r-2} + p_0 x r x^{r-1} + q_0 x^r = 0$

$[r(r-1) + p_0 r + q_0]x^r = 0$

only one term x^r

$\Rightarrow r(r-1) + p_0 r + q_0 = 0$

Solve for r etc

General Case

x = 0 is a regular singular value

When $p_n \neq 0$ and/or $q_n \neq 0$ for some $n > 0$ then we will assume the solution to

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$$

has the form

Guess

$$y(x) = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n},$$

an Euler solution multiplied by a power series.

If not Euler for regular singular value

Solution Procedure

Assuming $y(x) = x^r\sum_{n=0}^{\infty} a_nx^n$ we must determine:

1. the values of r, ← new for regular singular value
2. a recurrence relation for a_n,
3. the radius of convergence of $\sum_{n=0}^{\infty} a_nx^n$.

Ratio test

similar to ordinary but w/ r

$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right| \; ? \; 1$

See exam 2 answers

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_nx^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_nx^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

$$y'(x) = \sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}$$

$$y''(x) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2}$$

Example (2 of 8)

$$\begin{aligned} 0 &= 4xy'' + 2y' + y \\ &= 4x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2} + 2\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty}a_nx^{r+n} \\ &= \sum_{n=0}^{\infty}4(r+n)(r+n-1)a_nx^{r+n-1} + \sum_{n=0}^{\infty}2(r+n)a_nx^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty}a_nx^{r+n} \\ &= \sum_{n=0}^{\infty}[4(r+n)(r+n-1) + 2(r+n)]\,a_nx^{r+n-1} + \sum_{n=0}^{\infty}a_nx^{r+n} \end{aligned}$$

5.4: Euler equation: $x^2y'' + \alpha xy' + \beta y = 0$

Let $L(y) = x^2y'' + \alpha xy' + \beta y$

Recall that L is a linear function and if f is a solution to the euler equation, then $L(f) = 0$.

Note that if $x \neq 0$, then x is an ordinary point and if $x = 0$, then x is a singular point.

Suppose $x > 0$. Claim $L(x^r) = 0$ for some value of r

$y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2r(r-1)x^{r-2} + \alpha xrx^{r-1} + \beta x^r = 0$

$(r^2 - r)x^r + \alpha rx^r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$

$x^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus x^r is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Suppose $x < 0$. Claim $L((-x)^r) = 0$ for some value of r

$y = (-x)^r$, $y' = -r(-x)^{r-1}$, $y'' = r(r-1)(-x)^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2r(r-1)(-x)^{r-2} - \alpha xr(-x)^{r-1} + \beta(-x)^r = 0$

$(r^2 - r)(-x)^r + \alpha r(-x)^r + \beta(-x)^r = 0$

$(-x)^r[r^2 - r + \alpha r + \beta] = 0$

$(-x)^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus $(-x)^r$ is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Recall $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x|^r = \begin{cases} x^r & \text{if } x > 0 \\ (-x)^r & \text{if } x < 0 \end{cases}$

Thus if $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$, then $y = |x|^r$ is a solution to Euler's equation for $x \neq 0$.

for Euler eqn case

Case 1. 2 real distinct roots, r_1, r_2:
General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$.

Case 2: 2 complex solutions $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\lambda\pm i\mu} = e^{ln(|x|^{\lambda\pm i\mu})} = e^{(\lambda\pm i\mu)ln|x|} = e^{\lambda ln|x|}e^{i(\pm\mu ln|x|)}$

$= |x|^\lambda[cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)]$

$= |x|^\lambda[cos(\mu ln|x|) \pm isin(\mu ln|x|)]$

$|x|^\lambda \cdot |x|^{\pm i\mu}$

Case 3: 1 repeated root: Find 2nd solution. ?

Special Case: Euler's Equation

If $p_n = 0$ and $q_n = 0$ for $n \geq 1$ then

$$\begin{aligned} 0 &= x^2y'' + x\,[p_0 + p_1 x + \cdots + p_n x^n + \cdots]\,y' \\ &\quad + [q_0 + q_1 x + \cdots + q_n x^n + \cdots]\,y \\ &= x^2 y'' + p_0 x y' + q_0 y \end{aligned}$$

which is Euler's equation.

General Case

When $p_n \neq 0$ and/or $q_n \neq 0$ for some $n > 0$ then we will assume the solution to

$$x^2y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$$

has the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}$$

an Euler solution multiplied by a power series.

Solution Procedure

Assuming $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ we must determine:

1. the values of r,
2. a recurrence relation for a_n,
3. the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

Indicial eqn

$\frac{4}{2} r(r-1) + \frac{2}{2} r = \frac{0}{2}$

$r(2r-2+1) = 0$

$r(2r-1) = 0$

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} \\ y''(x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} \end{aligned}$$

Example (2 of 8)

$$\begin{aligned} 0 &= 4xy'' + 2y' + y \\ &= 4x \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} + 2\sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \sum_{n=0}^{\infty} 4(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} 2(r+n)a_n x^{r+n-1} \\ &\quad + \cancel{\sum_{n=0}^{\infty} a_n x^{r+n}} \quad + \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1} \\ &= \sum_{n=0}^{\infty} [4(r+n)(r+n-1) + 2(r+n)]\, a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \end{aligned}$$

$n \to n-1$

$= \sum 4(r+n)(r+n-1)$

$0 = [4r(r-1) [illegible] + 2r]a_0 x^{r-1} + \sum_{n=1}^{\infty} \left(2(r+n)a_n [2(r+n-1)+1] + a_{n-1}\right) x^{r+n-1}$

Example (3 of 8)

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty}[4(r+n)(r+n-1)+2(r+n)]\,a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1}
\end{aligned}
$$

$n \to n+1$, $n-1$

$$\sum_{n=-1}^{\infty} 2a_{n+1}(r+n+1)(2r+2(n+1)-1)x^{r+n}$$

$n+1$ goes from 0 to ∞

n goes from -1 to ∞

Example (4 of 8)

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1} \\
&= 2a_0 r(2r-1)x^{r-1} + \sum_{n=1}^{\infty} 2a_n(r+n)(2r+2n-1)x^{r+n-1} \\
&\quad + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1} \\
&= \underbrace{2a_0 r(2r-1)}_{=0}x^{r-1} + \sum_{n=1}^{\infty}\underbrace{[2a_n(r+n)(2r+2n-1)+a_{n-1}]}_{=0}x^{r+n-1}
\end{aligned}
$$

indicial eqn

recurrence relation

Example (5 of 8)

$$
\begin{aligned}
0 &= 2a_0 r(2r-1)x^{r-1} \\
&\quad + \sum_{n=1}^{\infty}[2a_n(r+n)(2r+2n-1)+a_{n-1}]\,x^{r+n-1}
\end{aligned}
$$

This implies

$$
\begin{aligned}
0 &= r(2r-1) \quad \text{(the \textbf{indicial equation}) and} \\
0 &= 2a_n(r+n)(2r+2n-1)+a_{n-1}
\end{aligned}
$$

Thus we see that $r = 0$ or $r = \frac{1}{2}$ and the recurrence relation is

$$a_n = -\frac{a_{n-1}}{(2r+2n)(2r+2n-1)}, \quad \text{for } n \ge 1.$$

Solve for a_* where $* =$ highest index

For this example solve for a_n

Example, Case $r = 0$ (6 of 8)

The recurrence relation becomes $a_n = -\frac{a_{n-1}}{2n(2n-1)}$.

$$
\begin{aligned}
a_1 &= -\frac{a_0}{(2)(1)} = -\frac{a_0}{2!} \\
a_2 &= -\frac{a_1}{(4)(3)} = \frac{a_0}{4!} \\
a_3 &= -\frac{a_2}{(6)(5)} = -\frac{a_0}{6!} \\
&\vdots \\
a_n &= \frac{(-1)^n a_0}{(2n)!}
\end{aligned}
$$

Thus $y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n a_0}{(2n)!}x^{n+0} = a_0\cos\sqrt{x}$.

~~Example, Case $r = 1/2$ (7 of 8)~~

~~The recurrence relation becomes $a_n = -\frac{a_{n-1}}{(2n+1)2n}$.~~

$$
\begin{aligned}
&\cancel{a_1 = -\frac{a_0}{(3)(2)} = -\frac{a_0}{3!}} \\
&\cancel{a_2 = -\frac{a_1}{(5)(4)} = \frac{a_0}{5!}} \\
&\cancel{a_3 = -\frac{a_2}{(7)(6)} = -\frac{a_0}{7!}} \\
&\vdots \\
&\cancel{a_n = \frac{(-1)^n a_0}{(2n+1)!}}
\end{aligned}
$$

Example, Case $r = 1/2$ (7 of 8)

The recurrence relation becomes $a_n = -\frac{a_{n-1}}{(2n+1)2n}$.

$$
\begin{aligned}
a_1 &= -\frac{a_0}{(3)(2)} = -\frac{a_0}{3!} \\
a_2 &= -\frac{a_1}{(5)(4)} = \frac{a_0}{5!} \\
a_3 &= -\frac{a_2}{(7)(6)} = -\frac{a_0}{7!} \\
&\vdots \\
a_n &= \frac{(-1)^n a_0}{(2n+1)!}
\end{aligned}
$$

Thus $y_2(x) = \sum_{n=0}^{\infty}\frac{(-1)^n a_0}{(2n+1)!}x^{n+1/2} = a_0\sin\sqrt{x}$.

$$y = c_1\left(\frac{(-1)^n x^n}{(2n)!}\right) + c_2\left(\frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!}\right)$$

Example (8 of 8)

We should verify that the general solution to

$$4xy'' + 2y' + y = 0$$

is

$$y(x) = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}.$$

Remarks

- This technique just outlined will succeed provided $r_1 \neq r_2$ and $r_1 - r_2 \neq n \in \mathbb{Z}$.
- If $r_1 = r_2$ or $r_1 - r_2 = n \in \mathbb{Z}$ then we can always find the solution corresponding to the larger of the two roots r_1 or r_2.
- The second (linearly independent) solution will have a more complicated form involving $\ln x$.

General Case: Method of Frobenius

Given $x^2 y'' + x\,[xp(x)]\,y' + \left[x^2 q(x)\right] y = 0$ where $x = 0$ is a regular singular point and

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

are analytic at $x = 0$, we will seek a solution to the ODE of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

where $a_0 \neq 0$.

Substitute into the ODE

$$\begin{aligned} 0 &= x^2 \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} \\ &\quad + x \left[\sum_{n=0}^{\infty} p_n x^n\right] \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + \left[\sum_{n=0}^{\infty} q_n x^n\right] \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n} \\ &\quad + \left[\sum_{n=0}^{\infty} p_n x^n\right] \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} + \left[\sum_{n=0}^{\infty} q_n x^n\right] \sum_{n=0}^{\infty} a_n x^{r+n} \end{aligned}$$

Collect Like Powers of x

$$\begin{aligned} 0 &= a_0 r(r-1)x^r + a_1(r+1)rx^{r+1} + \cdots \\ &\quad + (p_0 + p_1 x + \cdots)(a_0 r x^r + a_1(r+1)x^{r+1} + \cdots) \\ &\quad + (q_0 + q_1 x + \cdots)(a_0 x^r + a_1 x^{r+1} + \cdots) \\ &= a_0 \left[r(r-1) + p_0 r + q_0\right] x^r \\ &\quad + \left[a_1(r+1)r + p_0 a_1 (r+1) + p_1 a_0 r + q_0 a_1 + q_1 a_0\right] x^{r+1} \\ &\quad + \cdots \\ &= a_0 \left[r(r-1) + p_0 r + q_0\right] x^r \\ &\quad + \left[a_1\left((r+1)r + p_0(r+1) + q_0\right) + a_0\left(p_1 r + q_1\right)\right] x^{r+1} \\ &\quad + \cdots \end{aligned}$$

Indicial Equation

indicial eqn

If we define $F(r) = r(r-1) + p_0 r + q_0$ then the ODE can be written as

$$\begin{aligned} 0 &= a_0 F(r) x^r + \left[a_1 F(r+1) + a_0\left(p_1 r + q_1\right)\right] x^{r+1} \\ &\quad + \left[a_2 F(r+2) + a_0\left(p_2 r + q_2\right) + a_1\left(p_1(r+1) + q_1\right)\right] x^{r+2} \\ &\quad + \cdots \end{aligned}$$

The equation

$$0 = F(r) = r(r-1) + p_0 r + q_0$$

is called the **indicial equation**. The solutions are called the **exponents of singularity**.

Recurrence Relation

The coefficients of x^{r+n} for $n \geq 1$ determine the **recurrence relation**:

$$\begin{aligned} 0 &= a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left(p_{n-k}(r+k) + q_{n-k}\right) \\ a_n &= -\frac{\sum_{k=0}^{n-1} a_k \left(p_{n-k}(r+k) + q_{n-k}\right)}{F(r+n)} \end{aligned}$$

provided $F(r+n) \neq 0$.

indicial eqn evaluated at r+n

Exponents of Singularity

- By convention we will let the roots of the indicial equation $F(r) = 0$ be r_1 and r_2.
- When r_1 and $r_2 \in \mathbb{R}$ we will assign subscripts so that $r_1 \geq r_2$.
- Consequently the recurrence relation where $r = r_1$,

$$a_n(r_1) = -\frac{\sum_{k=0}^{n-1} a_k \left(p_{n-k}(r_1+k) + q_{n-k}\right)}{F(r_1+n)}$$

is defined for all $n \geq 1$.

- One solution to the ODE is then

$$y_1(x) = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1) x^n\right).$$

Case: $r_1 - r_2 \notin \mathbb{N}$

- If $r_1 - r_2 \neq n$ for any $n \in \mathbb{N}$ then $r_1 \neq r_2 + n$ for any $n \in \mathbb{N}$ and consequently $F(r_2 + n) \neq 0$ for any $n \in \mathbb{N}$.
- Consequently the recurrence relation where $r = r_2$,

$$a_n(r_2) = -\frac{\sum_{k=0}^{n-1} a_k \left(p_{n-k}(r_2+k) + q_{n-k}\right)}{F(r_2+n)}$$

is defined for all $n \geq 1$.

- A second solution to the ODE is then

$$y_2(x) = x^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2) x^n\right).$$

Example

Find the indicial equation, exponents of singularity, and discuss the nature of solutions to the ODE

$$x^2 y'' - x(2+x)y' + (2+x^2)y = 0$$

near the regular singular point $x = 0$.

Solution

$$\begin{aligned} p_0 &= \lim_{x\to 0} x\frac{-x(2+x)}{x^2} = -\lim_{x\to 0}(2+x) = -2 \\ q_0 &= \lim_{x\to 0} x^2\frac{2+x^2}{x^2} = \lim_{x\to 0}(2+x^2) = 2 \end{aligned}$$

The indicial equation is then

$$\begin{aligned} r(r-1) + (-2)r + 2 &= 0 \\ r^2 - 3r + 2 &= 0 \\ (r-2)(r-1) &= 0. \end{aligned}$$

The exponents of singularity are $r_1 = 2$ and $r_2 = 1$. Consequently we have one solution of the form

$$y_1(x) = x^2\left(1 + \sum_{n=1}^{\infty} a_n x^n\right).$$

Case: $r_1 = r_2$ Equal Exponents of Singularity (1 of 4)

- When the exponents of singularity are equal then $F(r) = (r - r_1)^2$.
- We have a solution to the ODE of the form

$$y_1(x) = x^r\left(1 + \sum_{n=1}^{\infty} a_n(r) x^n\right).$$

- Differentiating this solution and substituting into the ODE yields

$$\begin{aligned} 0 &= a_0 F(r) x^r \\ &\quad + \sum_{n=1}^{\infty}\left[a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left(p_{n-k}(r+k) + q_{n-k}\right)\right] x^{r+n} \\ &= a_0 (r - r_1)^2 x^r. \end{aligned}$$

when a_n solves the recurrence relation.