

[20] 1.) Solve $y'' - 6y' + 9y = 0$, $y(0) = 2$, $y'(0) = 4$.

$$r^2 - 6r + 9 = (r - 3)^2 = 0. \text{ Thus } r = 3$$

$$\text{General solution: } y = c_1 e^{3t} + c_2 t e^{3t}$$

$$y' = 3c_1 e^{3t} + c_2 (e^{3t} + 3t e^{3t})$$

$$y(0) = 2: \quad 2 = c_1$$

$$y'(0) = 4: \quad 4 = 3c_1 + c_2. \text{ Thus } c_2 = 4 - 6 = -2.$$

$$\text{Answer: } \underline{y = 2e^{3t} - 2te^{3t}}$$

2.) Circle T for true and F for false.

[4] 2a.) The equation $\ln(t)y' = \frac{t}{t+1} - y(\sin t^2)$ is a linear differential equation. T

[4] 2b.) The equation $y' + y = y^2$ is a linear differential equation. F

[4] 2c.) Suppose $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to $ay'' + by' + cy = 0$. If $y = h(t)$ is also a solution to $ay'' + by' + cy = 0$, then there exists constants c_1 and c_2 such that $h(t) = c_1 \phi_1(t) + c_2 \phi_2(t)$. F

[4] 2d.) Suppose $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $ay'' + by' + cy = 0$. If $y = h(t)$ is also a solution to $ay'' + by' + cy = 0$, then there exists constants c_1 and c_2 such that $h(t) = c_1 \phi_1(t) + c_2 \phi_2(t)$. T

[4] 3.) By giving a specific counter-example, prove that $y = \ln(x)$ is not a linear function.

Proof 1: $\ln(2) = \ln(1 + 1) \neq 0 = \ln(1) + \ln(1)$.

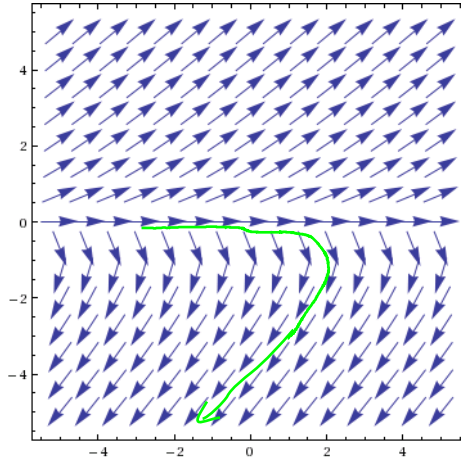
Proof 2: $\ln(e) + \ln(1) = 1 + 0 = 1$. But $\ln(e + 1) > \ln(e) = 1$ since $y = \ln(x)$ is an increasing function (since $[\ln(x)]' = \frac{1}{x} > 0$ for $x > 0$).

Proof 3: $\ln(e) + \ln(e) = 1 + 1 = 2$. But $\ln(e + e) = \ln(2e) = \ln(2) + \ln(e) = \ln(2) + 1$.

$\ln(2) < \ln(e) = 1$. Thus $2 \neq \ln(2) + 1$. Hence $\ln(e) + \ln(e) \neq \ln(2e)$ and thus $y = \ln(x)$ is not a linear function.

Proof 4: $e \cdot \ln(1) = e \cdot 0 = 0 \neq 1 = \ln(e) = \ln(e \cdot 1)$. In other words, if $f(x) = \ln(x)$, $f(cx) \neq cf(x)$ for constant $c = e$ and $x = 1$.

[20] 4.)



4a.) Circle the differential equation whose direction field is given above.

v.) $y' = \frac{y}{y+2}$

4b.) Draw the solution to the differential equation whose direction field is given above that satisfies the initial condition $y(1) = -3$

See above

4c.) Does the differential equation whose direction field is given above have any equilibrium solutions? If so, state whether they are stable, semi-stable or unstable.

The constant solution $y = 0$ is an unstable equilibrium solution.

[20] 5.) Solve: $y' = e^{4t} - \frac{y}{t}$

Linear, not separable: $y' + \frac{y}{t} = e^{4t}$

Let $u(t) = e^{\int \frac{dt}{t}} = e^{\ln|t|+C} = C|t|$. Let $u(t) = t$

$$ty' + y = te^{4t}$$

$$(ty)' = te^{4t}$$

$$\int (ty)' = \int te^{4t} dt.$$

$$\text{Let } u = t \quad dv = e^{4t}$$

$$du = dt \quad v = \frac{e^{4t}}{4}$$

$$ty = \frac{te^{4t}}{4} - \int \frac{e^{4t}}{4} dt = \frac{te^{4t}}{4} - \frac{e^{4t}}{16} + C$$

$$\text{Thus } y = \frac{e^{4t}}{4} - \frac{e^{4t}}{16t} + \frac{C}{t}$$

$$\text{Answer: } \underline{y = \frac{e^{4t}}{4} - \frac{e^{4t}}{16t} + \frac{C}{t}}$$

[20] 6.) Choose one of the following 2 problems. If you do not choose your best problem, I will substitute the other problem, but with a 2 point penalty (if it improves your grade). Circle the letter corresponding to your chosen problem: A B

6A.) Show that $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k t^k}{k!}$ converges for all t and show that $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k t^k}{k!}$ is a solution to $y' = 3t - y$.

Proof: $\lim_{k \rightarrow \infty} \frac{\frac{3(-1)^{k+1} t^{k+1}}{(k+1)!}}{\frac{3(-1)^k t^k}{k!}} = \lim_{k \rightarrow \infty} \frac{t}{k+1} = 0 < 1$ for all t .

Thus $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k t^k}{k!}$ converges for all t by the ratio test.

To show that $y = \phi(t)$ is a solution to $y' = 3t - y$, we plug it in.

$$\phi' = \sum_{k=2}^{\infty} \frac{3(-1)^k (k)t^{k-1}}{k!} = \sum_{k=2}^{\infty} \frac{3(-1)^k t^{k-1}}{(k-1)!}$$

$$3t - \phi = 3t - \sum_{k=2}^{\infty} \frac{3(-1)^k t^k}{k!} = 3t - \sum_{k=3}^{\infty} \frac{3(-1)^{k-1} t^{k-1}}{(k-1)!}$$

$$\begin{aligned} \phi' &= \sum_{k=2}^{\infty} \frac{3(-1)^k (k)t^{k-1}}{k!} = \sum_{k=2}^{\infty} \frac{3(-1)^k t^{k-1}}{(k-1)!} = \sum_{k=2}^2 \frac{3(-1)^k t^{k-1}}{(k-1)!} + \sum_{k=3}^{\infty} \frac{3(-1)^k t^{k-1}}{(k-1)!} \\ &= 3t - \sum_{k=3}^{\infty} \frac{3(-1)^{k-1} t^{k-1}}{(k-1)!} = 3t - \phi \end{aligned}$$

Thus $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k t^k}{k!}$ is a solution to $y' = 3t - y$.

6B.) Show by induction that for Picard's iteration method, $\phi_n(t) = \sum_{k=1}^n \frac{3(-1)^{k+1} t^{k+1}}{(k+1)!}$ approximates the solution to the initial value problem, $y' = 3t - y$, $y(0) = 0$ where $\phi_1(t) = \frac{3t^2}{2}$. You may use the proof outline below or write it from scratch.

Proof by induction on n .

$$\text{For } n = 1, \quad \sum_{k=1}^1 \frac{3(-1)^{k+1} t^{k+1}}{(k+1)!} = \frac{3(-1)^{1+1} t^{1+1}}{(1+1)!} = \frac{3 t^2}{2!} = \phi_1$$

$$\text{Suppose for } n = j - 1, \quad \phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{3(-1)^{k+1} t^{k+1}}{(k+1)!}$$

Then by Picard's iteration method, $\phi_j = \int_0^t f(s, \phi_{j-1}(s)) ds$ where $f(t, y) = 3t - y$.

$$\begin{aligned} \text{Thus } \phi_j &= \int_0^t (3s - \phi_{j-1}(s)) ds = \int_0^t \left(3s - \sum_{k=1}^{j-1} \frac{3(-1)^{k+1} s^{k+1}}{(k+1)!} \right) ds \\ &= \int_0^t 3s \, ds - \sum_{k=1}^{j-1} \int_0^t \frac{3(-1)^{k+1} s^{k+1}}{(k+1)!} ds \\ &= \frac{3}{2} s^2 \Big|_0^t - \sum_{k=1}^{j-1} \frac{3(-1)^{k+1} s^{k+2}}{(k+2)!} \Big|_0^t \\ &= \frac{3}{2} t^2 - 0 - \left(\sum_{k=1}^{j-1} \frac{3(-1)^{k+1} t^{k+2}}{(k+2)!} - 0 \right) \\ &= \frac{3}{2} t^2 + \sum_{k=1}^{j-1} \frac{3(-1)^{k+2} t^{k+2}}{(k+2)!} \\ &= \sum_{k=1}^1 \frac{3(-1)^{k+1} t^{k+1}}{(k+1)!} + \sum_{k=2}^j \frac{3(-1)^{k+1} t^{k+1}}{(k+1)!} \\ &= \sum_{k=1}^j \frac{3(-1)^{k+1} t^{k+1}}{(k+1)!} \end{aligned}$$