Defn: Let $f : \mathbf{R}^n \to \mathbf{R}^m$. Define $g_{ij} : \mathbf{R}^1 \to \mathbf{R}^1$, $g_{ij}(t) = f_i(x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n)$. If g is differentiable at a, then the partial derivative of f_i is defined by

$$\frac{\partial f_i}{\partial x_j}(\mathbf{a}) = lim_{h\to 0}\frac{f_i(a_1, ..., a_{j-1}, a_j + h, a_{j+1}, ..., a_n) - f_i(a_1, ..., a_{j-1}, a_j, a_{j+1}, ..., a_n)}{h}$$

$$= lim_{h\to 0}\frac{f_i(\mathbf{a} + h\mathbf{e_j}) - f_i(\mathbf{a})}{h}$$

Ex: $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = \begin{cases} 0 & \text{(x, y) = (0, 0)} \\ \frac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$

$$\frac{\partial f}{\partial x}(0, 0) = lim_{h\to 0}\frac{f(0+h, 0) - f(0, 0)}{h} = lim_{h\to 0}\frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = lim_{h\to 0}\frac{f(0, 0+h) - f(0, 0)}{h} = lim_{h\to 0}\frac{0-0}{h} = 0$$

BUT f is not continuous at (0, 0)!!!!!!!!!

Defn: Suppose $f : \mathbf{R}^1 \to \mathbf{R}^1$. The derivative of f at the point a is $f'(a) = lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$ if above limit exists.

$f'(a)$ is the derivative of f at x if

$$lim_{h\to 0}\frac{f(a+h) - f(a) - f'(a)h}{h} = 0$$

$$lim_{x\to a}\frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} = 0$$

$y = f'(a)[x - a] + f(a)$ is the linear approximation of f near a.

Defn: Let V and W be vector spaces. A **linear transformation** from V to W is a function $T : V \to W$ that satisfies the following two conditions. For each $\mathbf{u}$ and $\mathbf{v}$ in V and scalar a,

i.) $T(a\mathbf{u}) = aT(\mathbf{u})$

ii.) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

Thm: Let $T : V \to W$ be a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$

Pf: $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$

Thm: Let A be an $m \times n$ matrix. Then the function

$$T : \mathbf{R}^n \to \mathbf{R}^m$$
$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation.

Thm: If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = [T(\mathbf{e_1})...T(\mathbf{e_n})]$$

Ex: If $T : \mathbf{R}^2 \to \mathbf{R}$ and $T(1, 0) = 3$, $T(0, 1) = 4$, then

$$T(x, y) = xT(1, 0) + yT(0, 1) = (3, 4)\begin{pmatrix} x \\ y \end{pmatrix} = 3x + 4y$$

Defn: Suppose $f : \mathbf{R}^1 \to \mathbf{R}^1$ is differentiable at a. Then

$$lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$$

$$lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} = 0$$

Defn: Suppose $A \subset \mathbf{R}^n$, $f : A \to \mathbf{R}^m$.

f is said to be **differentiable at a point a** if there exists an open ball V such that $a \in V \subset A$ and a linear function T such that

$$lim_{\mathbf{h} \to \mathbf{0}} \frac{||f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})||}{||\mathbf{h}||} = 0$$

$$lim_{\mathbf{x} \to \mathbf{a}} \frac{||f(\mathbf{x}) - f(\mathbf{a}) - T(\mathbf{x}-\mathbf{a})||}{||\mathbf{x}-\mathbf{a}||} = 0$$

Suppose $f : \mathbf{R}^n \to \mathbf{R}$.

Then $T = (b_1, ..., b_n)$ and $T\mathbf{x} = (b_1, ..., b_n) \begin{pmatrix} x_1 \\ x_1 \\ ... \\ x_n \end{pmatrix} = \Sigma b_i x_i$

$$lim_{\mathbf{x} \to \mathbf{a}} \frac{||f(\mathbf{x}) - f(\mathbf{a}) - \Sigma b_i(x_i - a_i)||}{||\mathbf{x}-\mathbf{a}||} = 0$$

$f : \mathbf{R} \to \mathbf{R}$:

$$lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} = 0$$

$y = f'(a)[x-a] + f(a)$ is the linear approximation of f near a.

$\frac{df}{dx}(a) = f'(a)$

$f : \mathbf{R}^n \to \mathbf{R}$:

$$lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \Sigma b_i(x_i - a_i)}{||\mathbf{x}-\mathbf{a}||} = 0$$

$y = f(\mathbf{a}) + \Sigma b_i(x_i - a_i)$ approximates $y = f(\mathbf{x})$

$f(\mathbf{x}) = f(\mathbf{a}) + \Sigma b_i(x_i - a_i) + ||\mathbf{x}-\mathbf{a}||r(\mathbf{x}, \mathbf{a})$
where $lim_{\mathbf{x} \to \mathbf{a}} r(\mathbf{x}, \mathbf{a}) = 0$

$(df)_a = \Sigma b_i(x_i - a_i)$

Mean Value Theorem: Suppose
1.) f continuous on $[a, b]$
2.) f differentiable on (a, b)

Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

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