6.5 Another Forbidden Position Problem

Goal: To **derive** a formula for counting the number of permutations with relative forbidden positions.

Ex: Suppose children 1, 2, 3, 4, and 5 sit in a row in class. Children 1 and 2 cannot sit next to each other or they will cause trouble.

The order in which the children sit corresponds to a permutation of $\{1, 2, 3, 4, 5\}$. If 1 is in the ith spot, then 2 cannot be in the $i-1$st spot or the $i+1$th spot. Thus the pattern 21 or 12 cannot appear in our permutation. This is called a relative forbidden position as certain positions for the placement of 2 are forbidden, but these forbidden positions depend on the placement of 1.

We will focus on the relative forbidden position problem in which

Let $Q_n =$ the number of permutations of $\{1, 2, ..., n\}$ in which none of the patterns 12, 23, 34, ..., $(n-1)n$ occurs.

Thm 6.5.1 $Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - ... + \binom{n-1}{n-1}(-1)^{n-1}1!$

Proof: Use inclusion-exclusion principle.

Let $S =$ the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Let $A_j =$ set of permutations which contain the pattern $j(j+1)$ for $j = 1, ..., \mathbf{n-1}$.

To determine $|A_j|$, we can first look at all the permutations of $\{1, 2, ..., j, j+2, ..., n\}$. Since this set has $n-1$ elements, the number of permutations of $\{1, 2, ..., j, j+2, ..., n\} = (n-1)!$. Note that these permutations are in 1-1 correspondence with A_j as any permutation in A_j can be formed from a permutation of $\{1, 2, ..., j, j+2, ..., n\}$ by inserting $j+1$ right after j. Thus $|A_j| = (n-1)!$

Claim: $|A_i \cap A_j| = (n-2)!$ if $1 \leq i < j \leq n$

Suppose $|i-j| \geq 2$. In this case, $A_i \cap A_j$ is in 1-1 correspondence with permutations of a set of $n-2$ elements as we can create any permutation of $\{1, 2, ..., i, i+2, ..., j, j+2, ..., n\}$ by taking a permutation in $A_i \cap A_j$ and deleting $i+1$ and $j+1$. Thus $|A_j| = (n-2)!$ in this case.

Suppose $|i-j| = 1$. Thus $j = i+1$ (since we assumed $i < j$). $A_i \cap A_{i+1}$ is the set of permutations which contain the pattern $i\ i+1$ and the pattern $i+1\ i+2$. Thus $A_i \cap A_{i+1}$ is the set of permutations which contain the pattern $i\ i+1\ i+2$. Thus $A_i \cap A_{i+1}$ is in 1-1 correspondence with permutations of a set of $n-2$ elements as we can create any permutation of $\{1, 2, ..., i, i+3, ..., n\}$ by taking a permutation in $A_i \cap A_{i+1}$ and deleting $i+1$ and $i+2$. Thus $|A_i \cap A_{i+1}| = (n-2)!$ in this case as well.

Similarly, $|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = (n-k)!$.

Thus by inclusion-exclusion $Q_n = n! - \Sigma_{j=1}^{n-1}(n-1)! + \Sigma_{i,j}(n-2)! - ... + (-1)^n(n-n)!$

$$= \binom{n-1}{0}n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - ... + \binom{n-1}{n-1}(-1)^{n-1}1!$$