HW: Ch 7: 4, 9, 16

Defn: A recurrence relation is *linear* if

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + ... + a_k(n)h_{n-k} + b(n)$$

A recurrence relation has order k if $a_k \neq 0$

Ex: Derangement

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

$$D_n = nD_{n-1} + (-1)^n$$

Defn: A linear recurrence relation is *homogeneous* if $b = 0$

Defn: A linear recurrence relation has *constant coefficients* if the a_i's are constant.

7.2: linear homogeneous recurrence relation with constant coefficients:

$h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k} = 0$
where a_i are constants.

Suppose ϕ and ψ are solutions to the above recurrence relation, then

Claim 1: $c\phi$ is a solutions for any constant c

Claim 2: $\phi + \psi$ is a solutions for any constant c

Hence if ϕ_i are solutions, then $\Sigma c_i \phi_i$ is a solutions for any constants c_i.

Thm 7.2.1: Suppose a_i are constants and $q \neq 0$. Then q^n is a solution to

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k} = 0$$

iff q is a root of the polynomial equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - ... - a_k = 0$$

If this characteristic equation has k distinct roots, $q_1, q_2, ..., q_k$,

then $h_n = c_1 q_1^n + c_2 q_2^n + ... + c_k q_k^n$ is the general solution.

I.e, given any initial values for $h_0, h_1, ..., h_{k-1}$, there exists $c_1, c_2, ..., c_k$ such that $h_n = c_1 q_1^n + c_2 q_2^n + ... + c_k q_k^n$ satisfies the recurrence relation and the initial conditions.

Thm 7.2.2: Suppose q_i is an s_i-fold root of the characteristic equation. Then

$H_i(n) = c_1 q_i^n + c_2 n q_i^n + ... + c_{s_i} n^{s_i - 1} q_i^n$

is a solution to the recurrence relation.

If the characteristic equation has t distinct roots $q_1, ..., q_t$ with multiplicity $s_1, ..., s_t$, respectively, then

$h_n = H_1(n) + ...H_t(n)$ is a general solution.

Ex: Solve the recurrence relation, $h_n + h_{n-2} = 0$,
$h_0 = 3, h_1 = 5$

Ex: Solve the recurrence relation,
$h_n - 2h_{n-1} + 2h_{n-3} - h_{n-4} = 0$
$h_0 = 3, h_1 = 3, h_2 = 7, h_3 = 15$,

7.3: Non-homogeneous Recurrence Relations.

$h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k} = b$

Let $k(h) = h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k}$

Suppose ϕ is a solution to the recurrence relation $k(h) = 0$ and β is a solution to the recurrence relation $k(h) = b$.

Claim: $\phi + \beta$ is a solution to

Suppose $h_n - a_1 h_{n-1} = b$.

If b is a polynomial, guess β is a polynomial of the same degree.

If $b = d^n$, guess $\beta = pd^n$