Ex: Solve the recurrence relation: $h_n + h_{n-2} = 0$, $h_0 = 3, h_1 = 5$

Guess q^n is a solution.

$q^n + q^{n-2} = q^{n-2}(q^2 + 1) = 0$

$q^2 + 1 = 0$ implies $q = \pm i$

Thus the general solution is $h_n = c_1 i^n + c_2(-i)^n$ (i.e., this function satisfies the recurrence relation. Now need to find c_i's resulting in initial conditions.

$h_0 = 3$: $c_1 + c_2 = 3$ implies $c_2 = 3 - c_1$

$h_1 = 5$: $c_1 i - c_2 i = 5$ implies $-c_1 + c_2 = 5i$

$-c_1 + 3 - c_1 = 5i$. Thus $-2c_1 + 3 = 5i$

Hence $c_1 = \frac{3-5i}{2}$ and $c_2 = 3 - (\frac{3-5i}{2}) = \frac{3+5i}{2}$

$h_n = (\frac{3-5i}{2})i^n + (\frac{3+5i}{2})(-i)^n$ satisfies the recurrence relation and the initial conditions.

$h_{2j} = (\frac{3-5i}{2})i^{2j} + (\frac{3+5i}{2})(-i)^{2j} = 3(-1)^j$

$h_{2j+1} = (\frac{3-5i}{2})i^{2j+1} + (\frac{3+5i}{2})(-i)^{2j+1} = -5(i)^{2j+2} = 5(-1)^j$

Ex: Solve the recurrence relation, $h_n - 2h_{n-1} + 2h_{n-3} - h_{n-4} = 0$, $h_0 = 3, h_1 = 3, h_2 = 7, h_3 = 15$,

Guess q^n is a solution.

$q^n - 2q^{n-1} + 2q^{n-3} - q^{n-4} = q^{n-4}(q^4 - 2q^3 + 2q - 1) = 0$,

$(q-1)^3(q+1) = 0$

q = 1 , 1, 1, -1

Note $n^j(1)^n$, $j = 0, 1, 2$, is a solution to the recurrence relation:

$nq^n - 2(n-1)q^{n-1} + 2(n-3)q^{n-3} - (n-4)q^{n-4} = 0$

$2q^3 - 6q + 4q = 0$

$n^2q^n - 2(n-1)^2q^{n-1} + 2(n-3)^2q^{n-3} - (n-4)^2q^{n-4} = 0$

$n^2q^n - 2(n^2 - 2n + 1)q^{n-1} + 2(n^2 - 6n + 9)q^{n-3} - (n^2 - 8n + 16)q^{n-4} = 0$

$-2q^3 + 18q - 16 = 0$

$h_n = c_1(1)^n + c_2n(1)^n + c_3n^2(1)^n + c_4(-1)^n$

$h_0 = 3 = c_1 + c_4$

$h_1 = 3 = c_1 + c_2 + c_3 - c_4$

$h_2 = 7 = c_1 + 2c_2 + 4c_3 + c_4$

$h_3 = 15 = c_1 + 3c_2 + 9c_3 - c_4$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & -1 & 3 \\ 1 & 2 & 4 & 1 & 7 \\ 1 & 3 & 9 & -1 & 15 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 4 & 0 & 4 \\ 0 & 3 & 9 & -2 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 6 & 4 & 12 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Ex: $h_n = 4h_{n-1} + 3n - 10$, $h_0 = 8$