Suppose a multiset of size k must contain the following:
between two to four (inclusive) x's,
zero, one, two or five y's.

Find the number of multisets of size k.
"Long" method: list all possibilities
between two to four (inclusive) x's: $x^2 + x^3 + x^4$
zero, one, two or five y's: $y^0 + y^1 + y^2 + y^5$

$$(x^2 + x^3 + x^4)(y^0 + y^1 + y^2 + y^5)$$
$$= x^2y^0 + x^2y^1 + x^2y^2 + x^2y^5 + x^3y^0 + x^3y^1 + x^3y^2 + x^3y^5 + x^4y^0 + x^4y^1 + x^4y^2 + x^4y^5$$
$$= x^2y^0 + (x^2y^1 + x^3y^0) + (x^2y^2 + x^3y^1 + x^4y^0) + (x^3y^2 + x^4y^1) + x^4y^2 + x^2y^5 + x^3y^5 + x^4y^5$$

Let h_k = number of multisets of size k.

$h_0 =$, $h_1 =$, $h_2 =$, $h_3 =$, $h_4 =$,
$h_5 =$, $h_6 =$, $h_7 =$, $h_8 =$, $h_9 =$,
$h_k =$ $k > 9$

"Short" method:
between two to four (inclusive) x's: $x^2 + x^3 + x^4$
zero, one, two or five y's: $x^0 + x^1 + x^2 + x^5$

$$g(x) = (x^2 + x^3 + x^4)(x^0 + x^1 + x^2 + x^5)$$
$$= x^2x^0 + (x^2x^1 + x^3x^0) + (x^2x^2 + x^3x^1 + x^4x^0) + (x^3x^2 + x^4x^1) + x^4x^2 + x^2x^5 + x^3x^5 + x^4x^5$$
$$= x^2 + 2x^3 + 3x^4 + 2x^5 + x^6 + x^7 + x^8 + x^9$$

Suppose a multiset consisting of integers between 0 and 5 inclusive of size k must contain the following:

even number of 0's: $x^0 + x^2 + x^4 + ... = \frac{1}{1-x^2}$

odd number of 1's: $x^1 + x^3 + x^5 + ... = \frac{x}{1-x^2}$

three or four 2's: $x^3 + x^4 = x^3(1+x)$

the number of 3's is a multiple of five: $x^0+x^5+x^{10}+... = \frac{1}{1-x^5}$

btwn zero to four (inclusive) 4's: $x^0+x^1+x^2++x^3+x^4 = \frac{1-x^5}{1-x}$

zero or one 5: $x^0 + x^1 = 1 + x$

Find the number of multisets of size n.

Find the number of multisets of size 100.

$$g(x) = (x^0 + x^2 + x^4 + ...)(x^1 + x^3 + x^5 + ...)(x^3 + x^4)(x^0 + x^5 + x^10 + ...)(x^0 + x^1 + x^2 + +x^3 + x^4)(x^0 + x)$$

$$= \frac{1}{1-x^2} \frac{x}{1-x^2} x^3(1+x) \frac{1}{1-x^5} \frac{1-x^5}{1-x}(1+x)$$

$$= \frac{x^4}{(1-x)^3} = x^4 \Sigma_{k=0}^{\infty} \binom{3+k-1}{k} x^k = \Sigma_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^{k+4}$$

7.7 Exponential Generating Functions

Instead of using $1, x, x^2, ..., x^n, ...$

we can use $\frac{1}{0!}, \frac{x}{1!}, \frac{x^2}{2!}, ..., \frac{x^n}{n!}, ...$

Ex: The standard generating function of 1, 1, 1, ... is

$g(x) = 1 + x + x^2 + ... + x^n + ... = \frac{1}{1-x}$

The exponential generating function of 1, 1, 1 is

$g^{(e)}(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}, ... = e^x$

Ex: The standard generating function of 2, 3, 4, 5, 0, 0, ... is

$g(x) = 2 + 3x + 4x^2 + 5x^3$

The exponential generating function of 2, 3, 4, 5, 0, 0, ... is

$g^{(e)}(x) = 2 + 3x + 4\frac{x^2}{2!} + 5\frac{x^3}{3!}$

Ex: The standard generating function of $1, a, a^2, ...$ is

$g(x) = 1 + ax + a^2x^2 + ... + a^nx^n + ... = \frac{1}{1-ax}$

The exponential generating function of $1, a, a^2, ...$ is

$g^{(e)}(x) = 1 + ax + a^2\frac{x^2}{2!} + ... + a^n\frac{x^n}{n!}, ... = e^{ax}$

Suppose a permutation of size k must contain the following:

between two to four (inclusive) x's;

zero, one, two or five y's.

Find the number of possible permutations of size k.

"Long" method: list all possibilities

between two to four (inclusive) x's: $\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

zero, one, two or five y's: $\frac{y^0}{0!} + \frac{y^1}{1!} + \frac{y^2}{2!} + \frac{y^5}{5!}$

$$\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(\frac{y^0}{0!} + \frac{y^1}{1!} + \frac{y^2}{2!} + \frac{y^5}{5!}\right)$$

$$= \frac{x^2}{2!}\frac{y^0}{0!} + \left(\frac{x^2}{2!}\frac{y^1}{1!} + \frac{x^3}{3!}\frac{y^0}{0!}\right) + \left(\frac{x^2}{2!}\frac{y^2}{2!} + \frac{x^3}{3!}\frac{y^1}{1!} + \frac{x^4}{4!}\frac{y^0}{0!}\right)$$
$$+ \left(\frac{x^3}{3!}\frac{y^2}{2!} + \frac{x^4}{4!}\frac{y^1}{1!}\right) + \frac{x^4}{4!}\frac{y^2}{5!} + \frac{x^2}{2!}\frac{y^5}{5!} + \frac{x^3}{3!}\frac{y^5}{5!} + \frac{x^4}{4!}\frac{y^5}{5!}$$

$$= \frac{2!}{2!0!}\frac{x^2y^0}{2!} + \left(\frac{3!}{2!1!}\frac{x^2y^1}{3!} + \frac{3!}{3!0!}\frac{x^3y^0}{3!}\right) + \left(\frac{4!}{2!2!}\frac{x^2y^2}{4!} + \frac{4!}{3!1!}\frac{x^3y^1}{4!} + \frac{4!}{4!0!}\frac{x^4y^0}{4!}\right)$$
$$+ \left(\frac{5!}{3!2!}\frac{x^3y^2}{5!} + \frac{5!}{4!1!}\frac{x^4y^1}{5!}\right) + \frac{6!}{4!2!}\frac{x^4y^2}{6!} + \frac{7!}{2!5!}\frac{x^2y^5}{7!} + \frac{8!}{3!5!}\frac{x^3y^5}{8!} + \frac{9!}{4!5!}\frac{x^4y^5}{9!}$$

Let h_k = number of multisets of size k.

$h_0 =$, $h_1 =$, $h_2 =$, $h_3 =$,

$h_4 =$, $h_5 =$, $h_6 =$,

$h_7 =$, $h_8 =$, $h_9 =$, $h_k =$ $k > 9$

"Short" method:

between two to four (inclusive) x's: $\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

zero, one, two or five y's: $\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^5}{5!}$

$$g^{(e)}(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^5}{5!}\right)$$

$$= \frac{x^2}{2!}\frac{x^0}{0!} + \left(\frac{x^2}{2!}\frac{x^1}{1!} + \frac{x^3}{3!}\frac{x^0}{0!}\right) + \left(\frac{x^2}{2!}\frac{x^2}{2!} + \frac{x^3}{3!}\frac{x^1}{1!} + \frac{x^4}{4!}\frac{x^0}{0!}\right)$$
$$+ \left(\frac{x^3}{3!}\frac{x^2}{2!} + \frac{x^4}{4!}\frac{x^1}{1!}\right) + \frac{x^4}{4!}\frac{x^2}{5!} + \frac{x^2}{2!}\frac{x^5}{5!} + \frac{x^3}{3!}\frac{x^5}{5!} + \frac{x^4}{4!}\frac{x^5}{5!}$$

$$= \frac{1}{2!0!}x^2 + \left(\frac{1}{2!1!} + \frac{1}{3!0!}\right)x^3 + \left(\frac{1}{2!2!} + \frac{1}{3!1!} + \frac{1}{4!0!}\right)x^4$$
$$+ \left(\frac{1}{3!2!} + \frac{1}{4!1!}\right)x^5 + \frac{1}{4!5!}x^6 + \frac{1}{2!5!}x^7 + \frac{1}{3!5!}x^8 + \frac{1}{4!5!}x^9$$

$$= \frac{2!}{2!0!}\frac{x^2}{2!} + \left(\frac{3!}{2!1!} + \frac{3!}{3!0!}\right)\frac{x^3}{3!} + \left(\frac{4!}{2!2!} + \frac{4!}{3!1!} + \frac{4!}{4!0!}\right)\frac{x^4}{4!}$$
$$+ \left(\frac{5!}{3!2!} + \frac{5!}{4!1!}\frac{x^5}{5!}\right) + \frac{6!}{4!2!}\frac{x^6}{6!} + \frac{7!}{2!5!}\frac{x^7}{7!} + \frac{8!}{3!5!}\frac{x^8}{8!} + \frac{9!}{4!5!}\frac{x^9}{9!}$$

Thm 7.7.1: Let the multiset $S = \{n_1 \cdot a_1, n_2 \cdot a_2, ..., n_k \cdot a_k\}$
Let $h_n =$ the number of n-permutations of S.

$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) ... f_{n_k}(x)$ where

$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^{n_i}}{n_i!}$

Suppose a permutation consisting of integers between 0 and 2 inclusive of size k must contain the following:

even number of 0's

odd number of 1's

any number of 2's

Find the number of permutations of size k.

Find the number of permutations of size 100.

7.6: A geometry example

Thm 7.6.1: Let h_n = the number of ways of dividing a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals which do not intersect in the interior of the polygonal region. Define $h_1 = 1$. Then

$$h_n = \Sigma_{k=1}^{n-1} h_k h_{n-k}, \ n \geq 2$$

$$h_n = \frac{1}{n}\binom{2n-2}{n-1}, \ n \geq 1$$

$h_n = \Sigma_{k=1}^{n-1} h_k h_{n-k},\ n \geq 2,\ h_1 = 1,\ h_2 = h_1 h_1 = 1$

$h_3 = h_1 h_2 + h_2 h_1 = 2$, $h_4 = h_1 h_3 + h_2 h_2 + h_3 h_1 = 2 + 1 + 2 = 5$

Let $g(x) = 0 + h_1 x + h_2 x^2 + h_3 x^3 + ...$

Then $[g(x)]^2 = (h_1 x + h_2 x^2 + h_3 x^3 + ...)(h_1 x + h_2 x^2 + h_3 x^3 + ...)$

$= (h_1^2) x^2 + (h_1 h_2 + h_2 h_1) x^3 + (h_1 h_3 + h_2 h_2 + h_3 h_1) x^4 + ...$

$=$