HW 2

6.) The number of numbers with n distinct digits not containing 2 or 7 = the number of $n-$permutations of $\{0, 1, 3, 4, 5, 6, 8, 9\}$.

The number of numbers with ≥ 9 distinct digits = 0

The number of numbers with 8 distinct digits = 7P(7,7)

The number of numbers with 7 distinct digits = 7P(7,6)

The number of numbers with 6 distinct digits = 7P(7,5)

The number of numbers with 5 distinct digits = 7P(7,4)

The number of numbers with 4 distinct digits $> 6000 = 3 \cdot 7 \cdot 6 \cdot 5 = 630$

The number of numbers with 4 distinct digits > 5400, but $< 6000 = 1 \cdot 4 \cdot 6 \cdot 5 = 120$

Total: $7[P(7,7) + P(7,6) + P(7,5) + P(7,4)] + 750$

9.) 15 people around a (rotating) round table: $\frac{P(15,15)}{15}$

If B on A's right: $\frac{(15)(1)13!}{15}$

If B not on A's right: $\frac{P(15,15)}{15} - \frac{(15)(1)13!}{15} = 14! - 13! = 13!(14-1) = 13(13!)$

If B on A's left: $\frac{(15)(1)13!}{15}$

If B not on A's right nor on A's left: $\frac{P(15,15)}{15} - \frac{(15)(1)13!}{15} - -\frac{(15)(1)13!}{15} = 14! - 13! - 13! = 13!(14-1-1) =$ 12(13!)

11.) If the middle number = k, then there are k - 2 choices for the smallest number and (20 - k) - 1 choices for the largest number.

$\Sigma_{k=3}^{18}(k-2)((20-k)-1) = \Sigma_{k=3}^{18}(k-2)(19-k) = \Sigma_{k=3}^{18}(21k-(2)(19)-k^2) = 21\Sigma_{k=3}^{18}k - \Sigma_{k=3}^{18}(2)(19) - \Sigma_{k=3}^{18}k^2 = 21\Sigma_{k=3}^{18}k - \Sigma_{k=3}^{18}(2)(19) - \Sigma_{k=3}^{18}k^2 = 21$ [18(19)/2 - 1 - 2] - (2)(19)(18 - 2) - (18(19)(37)/6 - 1 - 4) = 21[9(19) - 3] - (2)(19)(16) - 3(19)(37) + 5 = 19[9(21) - 32 - 3(37)] - 63 + 5 = 19[189 - 32 - 111] - 58 = 19[46] - 58 = 816

Alternatively,

Suppose there exists at least one pair of consecutive integers in $\{n_1, n_2, n_3\}$. Let n_1 and n_2 be the consecutive numbers where n_1 is the smallest integer in a pair of consecutive integers. I.e, $\{n_1, n_2, n_3\}$ = $\{n_1, n_1 + 1, n_3\}$ where $1 \leq n_1 \leq 19$ and $n_3 \neq n_1 - 1$ (since n_1 is the smallest for any consecutive pair of integers) and $n_i \in \{1, 2, ..., 20\}$ $\forall i$ Suppose $n_1 = 1$, then $n_2 = 2$ and there are 18 ways to choose n_3. Suppose $n_1 > 1$, then $2 \leq n_1 \leq 19$ and there are 18 ways to choose n_1, 1 way to choose n_2, and 17 ways to choose n_3 (since can't choose $n_1 - 1$)

$C(20,3) - [18 + (18)(17)] = 20(19)(18)/6 - (18)(18) = 19(20)(3) - (18)(18) = 1140 - 324 = 816$

Alternatively, similar to problem 39. Suppose we think of the 20 numbers as 20 sticks. The number of ways of removing 3 sticks such that no two are consecutive is the same as the number of integral solutions to $x_1 + x_2 + x_3 + x_4 = 17$ where $x_1, x_4 \geq 0$ and $x_2, x_3 \geq 1$. This is the same as the number of solutions to $x_1 + y_2 + 1 + y_3 + 1 + x_4 = 17$ where $x_1, x_4 \geq 0$, $y_2 = x_2 - 1 \geq 1 - 1 = 0$, $y_3 = x_3 - 1 \geq 1 - 1 = 0$. This is the same as the number of solutions to $x_1 + y_2 + y_3 + x_4 = 15$ where $x_1, x_4, y_2, y_3 \geq 0$.

Hence by thm 3.5.1, the answer is $\binom{15+4-1}{15} = \binom{18}{15} = \frac{18(17)(16)}{6} = 51(16) = 816$

17) 6 indistinguishable rooks: 6!

6 distinguishable: (6!)(6!)

2 red and 4 blue: $\frac{(6!)(6!)}{(2!)(4!)}$

19a.) 5 red and 3 blue: $\frac{(8!)(8!)}{(5!)(3!)}$

b.) 8 rooks on 12 by 12 board:

Choose 8 of 12 columns: $C(12,8) = \frac{(12!)}{(8!)(4!)}$

Place one rook in these 8 columns in eight different rows: $P(12,8) = \frac{(12!)}{(4!)}$

Color the rooks: $\frac{(8!)}{(5!)(3!)}$

Choose 8 columns AND place one rook in these columns in different rows AND color the rooks:

$$\frac{(12!)}{(8!)(4!)}\frac{(12!)}{(4!)}\frac{(8!)}{(5!)(3!)} = \frac{(12!)(12!)}{(4!)(4!)(5!)(3!)} = \frac{(11!)(11!)}{(2!)(2!)(5!)(3!)} = \frac{(11!)(11)(9!)}{(2)(4!)(3!)} = \frac{(11!)(11)(9)(8)(7!)}{(2)(4)(3)(2)(3!)} = \frac{(11!)(11)(7!)}{4}$$

16) See class notes.

20) Circular permutations: $P(10,10)/10 = 9!$
circular permutations with 0 and 9 opposite: $P(8,4)P(4,4) =$(1) (8)(7)(6)(5) (1) (4)(3)(2)(1) $= 8!$

circular permutations with 0 and 9 NOT opposite: $P(10,10)/10$ - $P(8,4)P(4,4) = 9! - 8! = 8!(9-1) = 8(8!)$

Alternatively:
Place 0 at head of table: 1 choice
Place 9 anywhere else except opposite of 0: 8 choices
Place remaining 8 numbers in remaining 8 places: P(8, 8) choices.

Total number of choices: $8P(8,8) = 8(8!)$

21) Permutations of $\{1 \cdot A, 2 \cdot D, 1 \cdot R, 2 \cdot E, 3 \cdot S,\} = \frac{9!}{2!2!3!}$

8-permutations:

If permutation does not contain A: $\frac{(8!)}{(2!)(1!)(2!)(3!)}$

If permutation contains exactly one D: $\frac{(8!)}{(1!)(1!)(1!)(2!)(3!)}$

If permutation does not contain R: $\frac{(8!)}{(1!)(2!)(2!)(3!)}$

If permutation contains exactly one E: $\frac{(8!)}{(1!)(2!)(1!)(1!)(3!)}$

If permutation contains exactly two S's: $\frac{(8!)}{(1!)(2!)(1!)(2!)(2!)}$

Total number of permutations: $\frac{(8!)}{(2!)(1!)(2!)(3!)} + \frac{(8!)}{(1!)(1!)(1!)(2!)(3!)} + \frac{(8!)}{(1!)(2!)(2!)(3!)} + \frac{(8!)}{(1!)(2!)(1!)(1!)(3!)} + \frac{(8!)}{(1!)(2!)(1!)(2!)(2!)} = \frac{(7!)}{3} + \frac{2(7!)}{3} + \frac{(7!)}{3} + \frac{2(7!)}{3} + 7! = \frac{6(7!)}{3} + 7! = 3(7!)$

27) 5 rooks on 8 by 8 board:

Suppose a rook which is placed in the first column is also placed in the first row (i.e., the rook is placed in the (1, 1) square of the 8 x 8 board. Then the remaining 4 rooks must be placed in non-attacking positions in the last 7 columns and last 7 rows.

Choose 4 of these 7 columns: $C(7,4) = \frac{(7!)}{(4!)(3!)}$

Choose a 4-permutation from these 7 rows: $P(7,4) = \frac{(7!)}{(3!)}$

Thus if a rook is placed in the (1, 1) square, the number of ways to place the remaining 4 rooks in non-attacking positions is $\frac{(7!)}{(4!)(3!)} \frac{(7!)}{(3!)}$

Suppose a rook is placed in the 1st column, but not in the (1, 1) square. There are 7 choices for which row this rook is placed. Call this row the r^{th} row.

Suppose a second rook is placed in the 1st row, but not in the (1, 1) square. There are 7 choices for which column this rook is placed. Call this column the c^{th} column.

Thus there are 49 choices for placing the first two rooks.

The remaining 3 rooks can be placed in any of the columns except the first column and the c^{th} column, leaving 6 columns from which to choose. Choose 3 of these 6 columns: $C(6,3) = \frac{(6!)}{(3!)(3!)}$

These 3 rooks can be placed in any of the rows except the first row and the r^{th} row, leaving 6 rows in which to place the rooks. Placing the 3 rooks in the 3 chosen columns and in 3 of these 6 rows = choosing a 3-permutation from these 6 rows: $P(6,3) = \frac{(6!)}{(3!)}$

Thus the number of ways to place the first two rooks AND to also place the remaining 3 rooks: $(49)\frac{(6!)}{(3!)(3!)}\frac{(6!)}{(3!)}$

Thus the total number of ways to place 5 rooks in an 8 by 8 board so that there is a rook in the first column and first row: $\frac{(7!)}{(4!)(3!)}\frac{(7!)}{(3!)} + (49)\frac{(6!)}{(3!)(3!)}\frac{(6!)}{(3!)}$

38) The number of integral solutions to $x_1 + x_2 + x_3 + x_4 = 30$ where $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, $x_4 \geq 8$ is the same number of solutions as $(y_1 + 2) + y_2 + (y_3 - 5) + (y_4 + 8) = 30$ where $y_1 = x_1 - 2 \geq 2 - 2$, $y_2 = x_2 \geq 0$, $y_3 = x_3 + 5 \geq -5 + 5$, $y_4 = x_4 - 8 \geq 8 - 8$ which has the same number of solutions as $y_1 + y_2 + y_3 + y_4 = 25$ where $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, $y_4 \geq 0$.

Hence the number of integral solutions to $x_1 + x_2 + x_3 + x_4 = 30$ where $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, $x_4 \geq 8 = \binom{25+4-1}{25} = \binom{28}{25}$

39a) C(20, 6)

b.) Similar to 11 (but choosing 6 instead of 3)

Suppose the i_1th, i_2th, i_3th, i_4th, i_5th, i_6th sticks were removed.

Let $x_1 = i_1 - 1 =$ number of sticks before the i_1th stick.

Let $x_j = i_j - i_{j-1} - 1$, for $j = i_2, ..., i_6 =$ number of sticks between the jth and (j + 1)th sticks.

Let $x_7 = 20 - i_6$ =the number of sticks after the i_6th stick.

The number of ways of removing 6 sticks from 20 such that no two are consecutive is the same as the number of integral solutions to $x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 14$ where $x_1, x_7 \geq 0$ and $x_i \geq 1$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1+y_2+1+y_3+1+y_4+1+y_5+1+y_6+1+x_7 = 14$ where $x_1, x_7 \geq 0$, $y_2 = x_2 - 1 \geq 1 - 1 = 0$, $y_i = x_i - 1 \geq 1 - 1 = 0$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1 + y_2 + y_3 + y_4 + y_5 + y_6 + x_7 = 9$ where $x_1, x_7, y_i \geq 0$ for $i = 2, 3, 4, 5, 6$.

Hence by thm 3.5.1, the answer is $\binom{9+7-1}{9} = \binom{15}{9}$

c.) In this case we need to know the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 14$ where $x_1, x_7 \geq 0$ and $x_i \geq 2$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 + y_6 + 2 + x_7 = 14$ where $x_1, x_7 \geq 0$, $y_2 = x_2 - 1 \geq 1 - 1 = 0$, $y_i = x_i - 2 \geq 2 - 2 = 0$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1 + y_2 + y_3 + y_4 + y_5 + y_6 + x_7 = 4$ where $x_1, x_7, y_i \geq 0$ for $i = 2, 3, 4, 5, 6$.

Hence by thm 3.5.1, the answer is $\binom{4+7-1}{4} = \binom{10}{4}$