

## Linear Functions

A function  $f$  is linear if  $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently  $f$  is linear if

1.)  $f(a\mathbf{x}) = af(\mathbf{x})$  and 2.)  $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If  $f$  is linear, then  $f(\mathbf{0}) = \mathbf{0}$

Proof:  $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1.)  $f : R \rightarrow R, f(x) = 2x$

Proof:

$$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$$

Example 2.)  $f : R^2 \rightarrow R^2,$   
 $f((x_1, x_2)) = (2x_1, x_1 + x_2)$

Proof: Let  $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$

$$a\mathbf{x} + b\mathbf{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2) = \blacksquare$$
$$(ax_1 + by_1, ax_2 + by_2)$$

$$f(ax_1 + by_1, ax_2 + by_2)$$

$$= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2)$$

$$= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2)$$

$$= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2)$$

$$= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2)$$

$$= af((x_1, x_2)) + bf((y_1, y_2))$$

Example 3.)  $D$  : set of all differential functions  $\rightarrow$  set of all functions,  $D(f) = f'$

Proof:

$$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD(g)$$

Example 4.) Given  $a, b$  real numbers,

$I$  : set of all integrable functions on  $[a, b] \rightarrow R$ ,

$$I(f) = \int_a^b f$$

Proof:  $I(sf + tg) = \int_a^b sf + tg = s \int_a^b f + t \int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose  $f^{-1}(x) = c$ ,  $f^{-1}(y) = d$ .

Then  $f(c) = x$  and  $f(d) = y$  and  
 $f(ac + bd) = af(c) + bf(d) = ax + by$ .

Hence  $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$ .

Example 6.)  $D$  : set of all twice differential functions  
 $\rightarrow$  set of all functions,  $L(f) = af'' + bf' + cf$

Proof:

$$\begin{aligned} L(sf + tg) &= a(sf + tg)'' + b(sf + tg)' + c(sf + tg) \\ &= saf'' + tag'' + sbf' + tbg' + scf + tcg \\ &= s(af'' + bf' + cf) + t(ag'' + bg' + cg) \\ &= sL(f) + tL(g) \end{aligned}$$

Consequence 1: If  $\psi_1, \psi_2$  are solutions to  $af'' + bf' + cf = 0$ , then  $3\psi_1 + 5\psi_2$  is also a solution to  $af'' + bf' + cf = 0$ ,

Proof: Since  $\psi_1, \psi_2$  are solutions to  $af'' + bf' + cf = 0$ ,  $L(\psi_1) = 0$  and  $L(\psi_2) = 0$ .

$$\begin{aligned} \text{Hence } L(3\psi_1 + 5\psi_2) &= 3L(\psi_1) + 5L(\psi_2) \\ &= 3(0) + 5(0) = 0. \end{aligned}$$

Thus  $3\psi_1 + 5\psi_2$  is also a solution to  $af'' + bf' + cf = 0$

Consequence 2:

If  $\psi_1$  is a solution to  $af'' + bf' + cf = h$   
and  $\psi_2$  is a solution to  $af'' + bf' + cf = k$ ,  
then  $3\psi_1 + 5\psi_2$  is a solution to  $af'' + bf' + cf = 3h + 5k$ ,

Since  $\psi_1$  is a solution to  $af'' + bf' + cf = h$ ,  $L(\psi_1) = h$ .

Since  $\psi_2$  is a solution to  $af'' + bf' + cf = k$ ,  $L(\psi_2) = k$ .

$$\begin{aligned} \text{Hence } L(3\psi_1 + 5\psi_2) &= 3L(\psi_1) + 5L(\psi_2) \\ &= 3h + 5k. \end{aligned}$$

Thus  $3\psi_1 + 5\psi_2$  is also a solution to  
 $af'' + bf' + cf = 3h + 5k$