$s'(t) = v(t)$

$= 3x^4 + 2 + \frac{3}{x} + \cos x$

$s(1) = 0$

$\Rightarrow$

$s(t) = \frac{3x^5}{5} + 2x + 3\ln|x|$

$+ \sin x + C$

$3x^{-1} \longrightarrow \cancel{\frac{3x^0}{0}}$

$\hookrightarrow 3\ln|x|$

$s(1) = 0$

$$0 = \frac{3(1)^5}{5} + 2(1) + 3\ln(1) + \sin(1) + C$$

$$0 = \frac{3}{5} + 2 + 0 + \sin(1) + C$$

$$0 = \frac{13}{5} + \sin(1) + C$$

$$\Rightarrow C = -\frac{13}{5} - \sin(1)$$

$$\boxed{s(t) = \frac{3x^5}{5} + 2x + 3\ln|x| + \sin x - \frac{13}{5} - \sin(1)}$$

Check:

$$s'(t) = 3x^4 + 2 + \frac{3}{x} + \cos x$$

$$s(1) = \frac{3}{5} + 2 + 0 + \sin(1)$$

$$-\frac{13}{5} - \sin(1) = 0$$

$$s(2) = ?$$

$$s(\pi) = ?$$

Biology application: Suppose the number of bacteria grow at an average rate or r = 10% per hour. If the initial population is 100 bacteria, find the number of bacteria after 1 hour, after 2 hours, after T hours.

Identical application, but in Finance:

Let $P(t)$ = amount in an account at time t (in years).

Ex 1: Suppose \$100 is deposited in the account earning an interest rate of r = 10% per year. Find the amount in the account after 1 year, after 2 years, after 3 years, and after T years.

$t = 0$: $P(0) = \$100$

$t = 1$: $P(1) = 100(1 + 0.1) = 100(1.1) = \$110$

$t = 2$: $P(2) = 100(1+0.1)(1+0.1) = 100(1+0.1)^2 = 100(1.1)^2 = \$121$

$t = 3$: $P(3) = 100(1 + 0.1)^3 = \$100(1.1)^3 = 133.10$

$\vdots$

$t = T$: $P(T) = 100(1 + 0.1)^T = \$100(1.1)^T$

The average interest rate earned is 10% per year.

The average rate of change in the account btwn year 0 and year 1:

$$\frac{P(1)-P(0)}{1} = 100(1.1) - 100 = 100(0.1) = \$10/\text{year}.$$

10%

The average rate of change between year t and year $t + 1$:

$$\frac{P(t+1)-P(t)}{1} = 100(1.1)^{t+1} - 100(1.1)^t = \underline{\$100(1.1)}^t(0.1)/\text{year}.$$

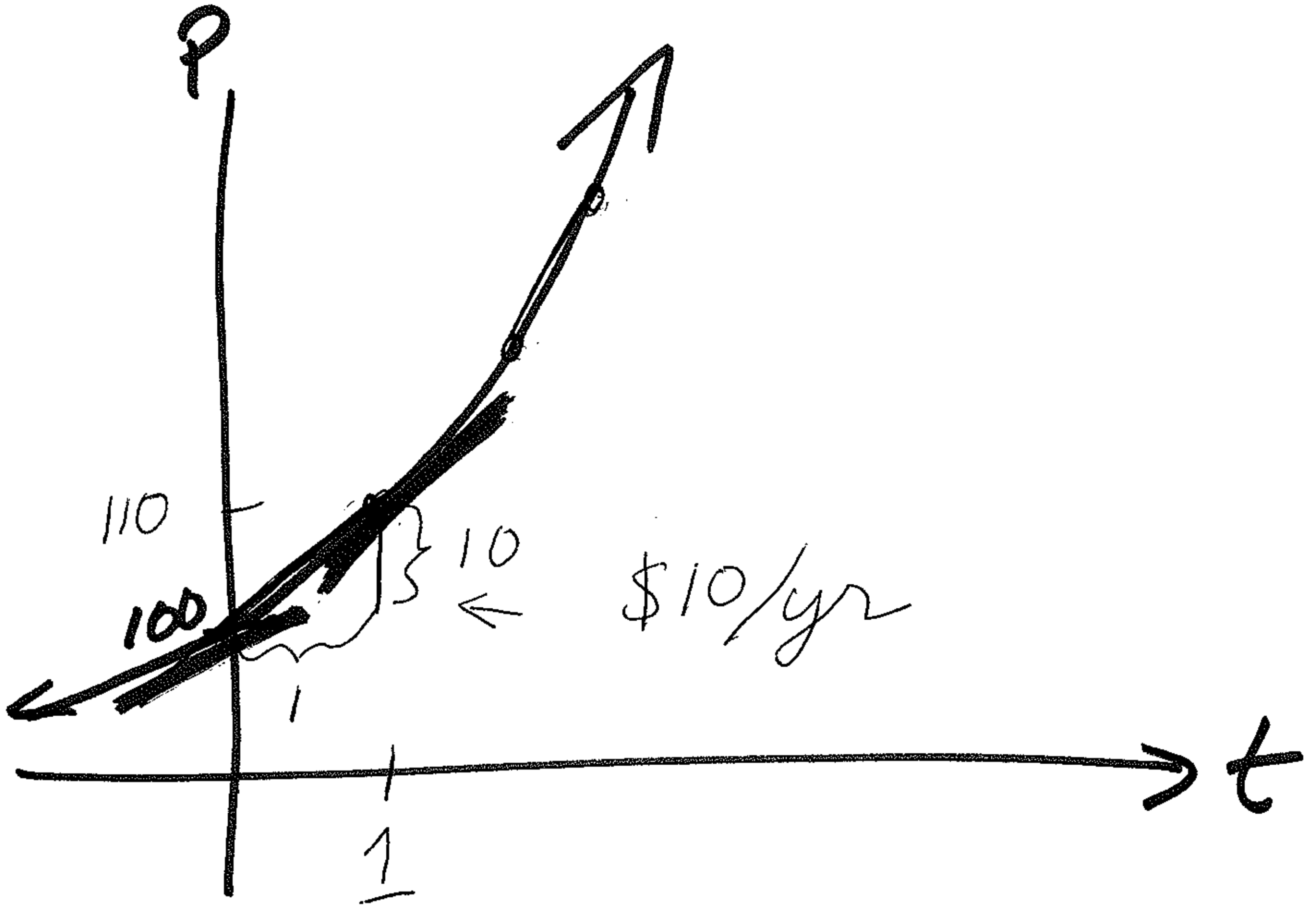
10% increase from year t

Instantaneous rate of change at time t:

$$P'(t) = [100(1.1)^t]' = 100 ln(1.1)(1.1)^t = (9.53102...) \cdot (1.1)^t$$

At $t = 1$: $P'(1) = 100 ln(1.1)(1.1) = \$10.48../yr$

\$/year


P
110
100
10
1
$10/yr
t
1

Ex 2: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded monthly. Find the amount in the account after 1 month, 1 year, 2 years, and T years.

$t = 0$: $P(0) = \$100$

$t = 1$ month: $P(\frac{1}{12}) = 100(1 + \frac{0.1}{12}) = \$100.83$

$t = 1$ year: $P(1) = 100(1 + \frac{0.1}{12})^{12} = \$110.47$

$t = 2$ years: $P(2) = 100(1 + \frac{0.1}{12})^{12\cdot 2} = \$122.04$

$\vdots$

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{12})^{12\cdot T} = \$100(1.1047...)^T$

The average interest rate earned is $\frac{10}{12}\%$ per month.

The average interest rate earned is 10.47...% per year.

The average rate of change between year t and year $t+1$:

$$\frac{P(t+1)-P(t)}{1} = 100(1+\tfrac{0.1}{12})^{12(t+1)} - 100(1+\tfrac{0.1}{12})^{12t}$$
$$= \$100(1+\tfrac{0.1}{12})^{12t}[(1+\tfrac{0.1}{12})^{12} - 1]/\text{year}.$$

The approximate average rate of change between year t and year $t+1$:

$$\frac{P(t+1)-P(t)}{1} = 100(1.1047)^{t+1} - 100(1.1047)^t$$
$$= \$100(1.1047)^t(0.1047)/\text{year}.$$

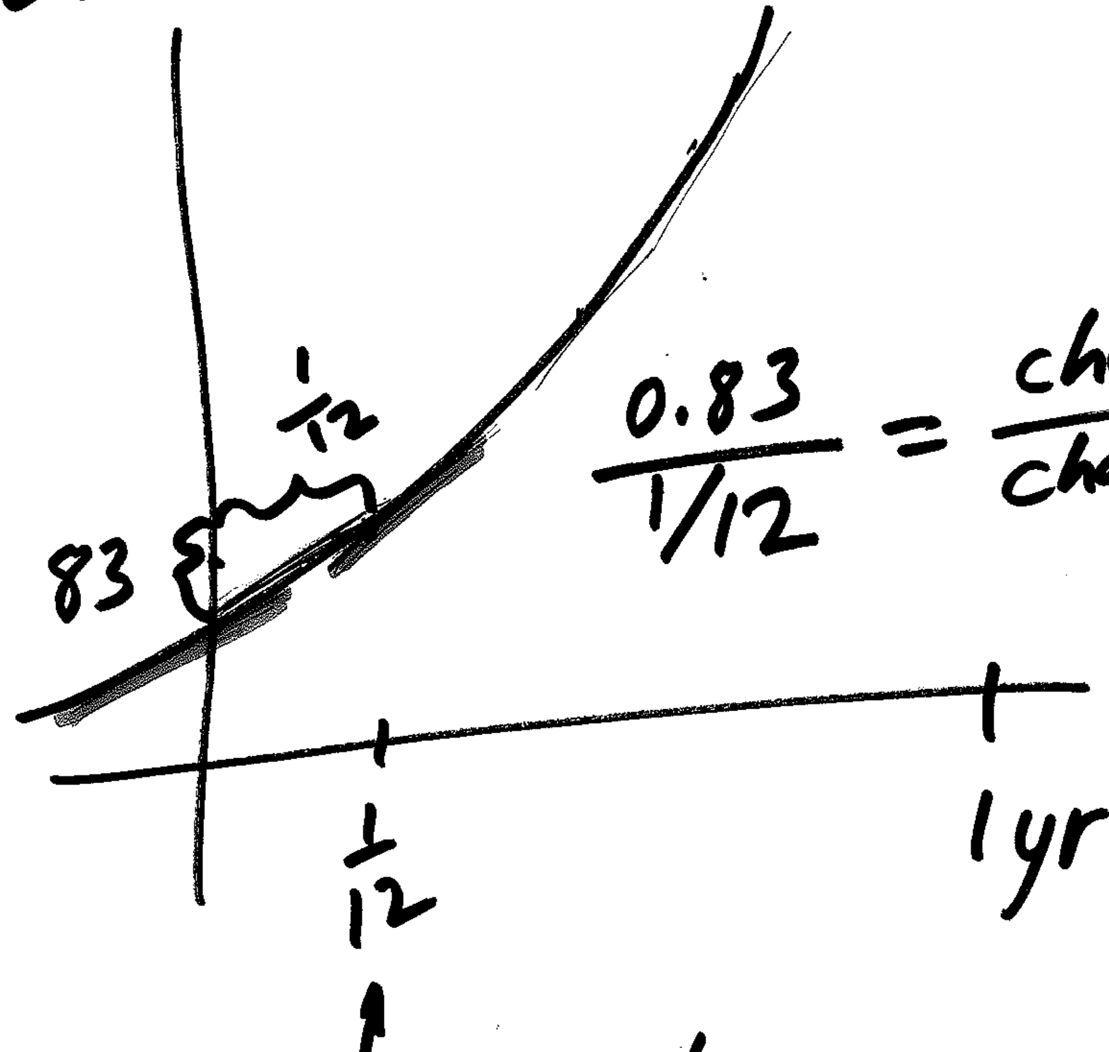
The instantaneous rate of change at time t:

$$P'(t) = [100(1+\tfrac{0.1}{12})^{12\cdot t}]' = 100 ln[(1+\tfrac{0.1}{12})^{12}]\cdot[(1+\tfrac{0.1}{12})^{12}]^t$$
$$= 1200 ln(1+\tfrac{0.1}{12})\cdot(1+\tfrac{0.1}{12})^{12t}$$
$$= \$(9.95856...)\cdot(1+\tfrac{0.1}{12})^{12t} / \text{year}$$

At $t = 1$: $P'(1) = 1200 ln(1+\frac{0.1}{12})\cdot(1+\frac{0.1}{12})^{12} = 11.001...$

At $t = \frac{1}{12}$, $P'(\frac{1}{12}) = 1200 ln(1+\frac{0.1}{12})\cdot(1+\frac{0.1}{12}) = 10.0416$

Ex2



$$\frac{0.83}{1/12} = \frac{\text{chang in \$}}{\text{change in time}}$$

1 month $= \frac{1}{12}$ yr

Ex 3: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded daily. Find the amount in the account after 1 day, 1 year, 2 years, and T years.

$t = 0$: $P(0) = \$100$

$t = 1$ day: $P(\frac{1}{356}) = 100(1 + \frac{0.1}{365}) = \$100.03$

$t = 1$ year: $P(1) = 100(1 + \frac{0.1}{365})^{365} = \$110.52$

$t = 2$ years: $P(2) = 100(1 + \frac{0.1}{365})^{365 \cdot 2} = \$122.14$

$\vdots$

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{365})^{365 \cdot T} = \$100(1.10515578...)^T$

The average interest rate earned is $\frac{10}{365}\%$ per day.

The average interest rate earned is 10.515578...% per year.

The average rate of change between year t and year $t + 1$:

$$\begin{aligned}\frac{P(t+1) - P(t)}{1} &= 100(1 + \tfrac{0.1}{365})^{365(t+1)} - 100(1 + \tfrac{0.1}{365})^{365t} \\ &= \$100(1 + \tfrac{0.1}{365})^{365t}[(1 + \tfrac{0.1}{365})^{365} - 1]/\text{year}.\end{aligned}$$

The instantaneous rate of change at time t:

$$\begin{aligned}P'(t) = [100(1 + \tfrac{0.1}{365})^{365 \cdot t}]' &= 100ln[(1 + \tfrac{0.1}{365})^{365}] \cdot [(1 + \tfrac{0.1}{365})^{365}]^t \\ &= 36500ln(1 + \tfrac{0.1}{365}) \cdot (1 + \tfrac{0.1}{365})^{365t} \\ &= (9.99863...) \cdot (1 + \tfrac{0.1}{365})^{365t}\end{aligned}$$

At $t = 1 : P'(1) = 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365})^{365} = 11.05...$

At $t = \frac{1}{365}$, $P'(\frac{1}{365}) = 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365}) = 10.00...$

Ex 4: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded n times per year. Find the amount in the account after T years.

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{n})^{n \cdot T}$

Ex 5: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded continuously. Find the amount in the account after T years.

$t = T$ years: $P(T) = lim_{n\to\infty} 100[(1 + \frac{0.1}{n})^{n \cdot T}] = 100e^{0.1T}$

Definition $e = lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.7...$

$\boxed{\text{FYI}}$: By Taylor series approximation from Calculus II

$$e^{0.1T} = lim_{n\to\infty}(1 + \tfrac{1}{n})^{n(0.1)T} = lim_{n\to\infty}[(1 + \tfrac{1}{n})^{0.1}]^{nT}$$

$$= lim_{n\to\infty}[1 + \tfrac{0.1}{n} - \tfrac{0.045}{n^2} + \tfrac{0.0285}{n^3} - ...]^{nT} = lim_{n\to\infty}[1 + \tfrac{0.1}{n}]^{nT}$$

Ex 6: Suppose $\$P_0$ is deposited in the account earning an interest rate of r ~~= s% per year ($r = \frac{s}{100}$), compounded continuously~~.

t years: $\boxed{P(t) = lim_{n\to\infty} P_0(1 + \frac{r}{n})^{n \cdot t} = P_0 e^{rt}}$

Ex 7: Suppose \$1 is deposited in the account earning an interest rate of $r = 10\%$ per year ($r = \frac{10}{100} = .1$), compounded continuously.

t years: $P(t) = lim_{n\to\infty}(1 + \frac{.1}{n})^{n \cdot t} = e^{0.1t}$

10% of $P(t)$

Note the instantaneous rate of change is 10% = $0.1e^{0.1t}$

That is $P'(t) = [e^{0.1t}]' = 0.1e^{0.1t}$

Ex 8: Suppose \$1 is deposited in the account earning an interest rate of $r = 100\%$ per year ($r = \frac{100}{100} = 1$), compounded continuously.

t years: $P(t) = lim_{n\to\infty}(1 + \frac{1}{n})^{n \cdot t} = e^t$

100%

Note the instantaneous rate of change is 100% = e^t

$\boxed{\text{That is } P'(t) = [e^t]' = e^t}$

Proof $(e^t)' = e^t = 100\%$ of e^t

4.5 $(e^x)' = e^x$

Find $(a^x)'$

Let $y = a^x$

Find y'

$$\ln y = \ln a^x$$

$$\ln y = x \cdot \ln a$$

$$\frac{d(\ln y)}{dx} = \frac{d(x \cdot \ln a)}{dx}$$

$$\frac{y'}{y} = \ln a$$

$$y' = y \ln a$$

$$y' = a^x \cdot \ln a$$

$$\boxed{(a^x)' = a^x \cdot \ln a}$$

$$(e^x)' = e^x \ln e = e^x$$

Alternate method

$$a^x = e^{\ln(a^x)} = e^{x \ln a}$$

$$(a^x)' = \left(e^{x \ln a}\right)'$$

$$= \left[e^{x \ln a}\right] \cdot [\ln a]$$

$$= e^{\ln a^x} \cdot [\ln a]$$

$$= (a^x)(\ln a)$$

Ex: $(2^x)' = 2^x \cdot \ln(2)$

$$\left(2^{[x^2]}\right)' = \left[2^{x^2} \cdot \ln 2\right] \cdot (2x)$$

3a

Find $(\log_a x)'$

$$y = \log_a x$$

$$a^y = a^{\log_a x}$$

$$a^y = x$$

$$\frac{d(a^y)}{dx} = \frac{dx}{dx}$$

$$[a^y \cdot \ln a] \cdot \frac{dy}{dx} = 1$$

$$(x \ln a)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{x \ln a}$$

/4

Alternate method

$$a^{\log_a x} = x$$

$$\ln\left(a^{\log_a x}\right) = \ln x$$

$$(\log_a x)(\ln a) = \ln x$$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{x \ln a}$$

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$$