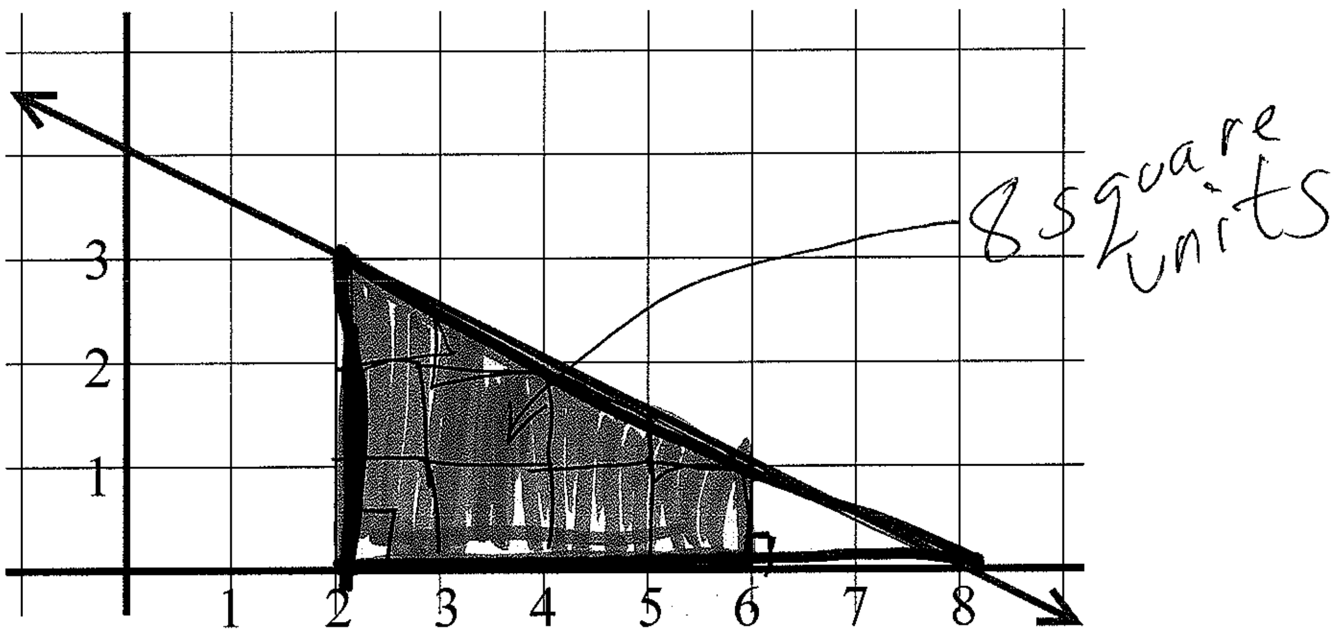
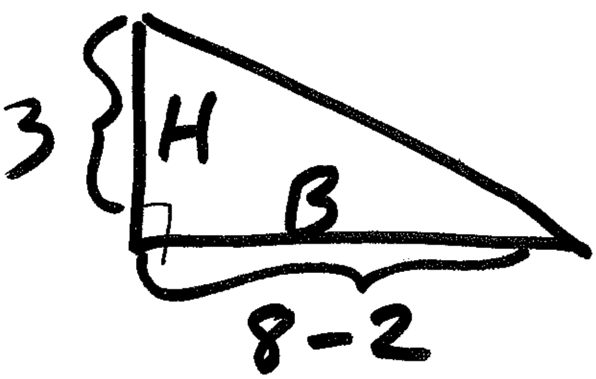
Section 5.2

Find the area under the curve $f(x) = -\frac{1}{2}x + 4$, above the $x-$axis and between $x = 2$ and $x = 6$.



Method 1: In this case our function is very simple, so we can determine the area without calculus:



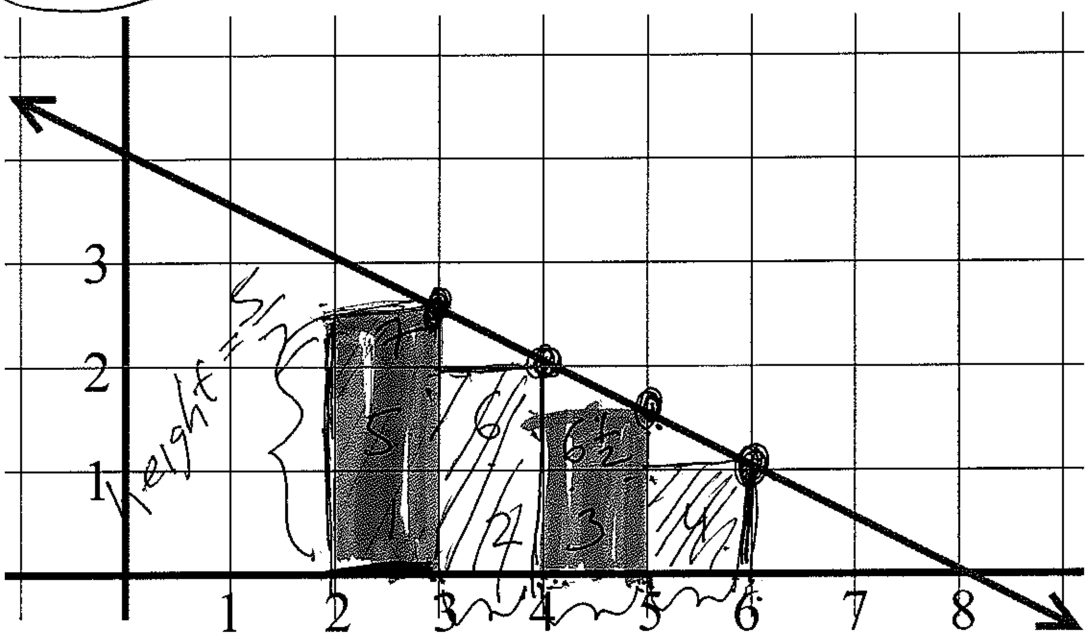
$$\frac{1}{2}BH - \frac{1}{2}bh = \frac{1}{2}(8-2)(3) - \frac{1}{2}(2)(1)$$

$$= \frac{1}{\cancel{2}}\cancel{6}^{3}\cdot 3 - \frac{1}{\cancel{2}}\cancel{2}\cdot 1 = 9 - 1 = 8$$

under-estimate

Method 2: Estimate using rectangles.

Inscribed rectangles with $\Delta x = 1$:



width = 1

$$\sum f(x_i)\,\Delta x = \sum_{i=3}^{6} f(i)(1) = \sum_{i=1}^{4} f(i+2)(1)$$

width

$$f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) =$$

height width height

$$= [-\tfrac{1}{2}(3) + 4](1) + [-\tfrac{1}{2}(4) + 4](1) + [-\tfrac{1}{2}(5) + 4](1) + [-\tfrac{1}{2}(6) + 4](1)$$

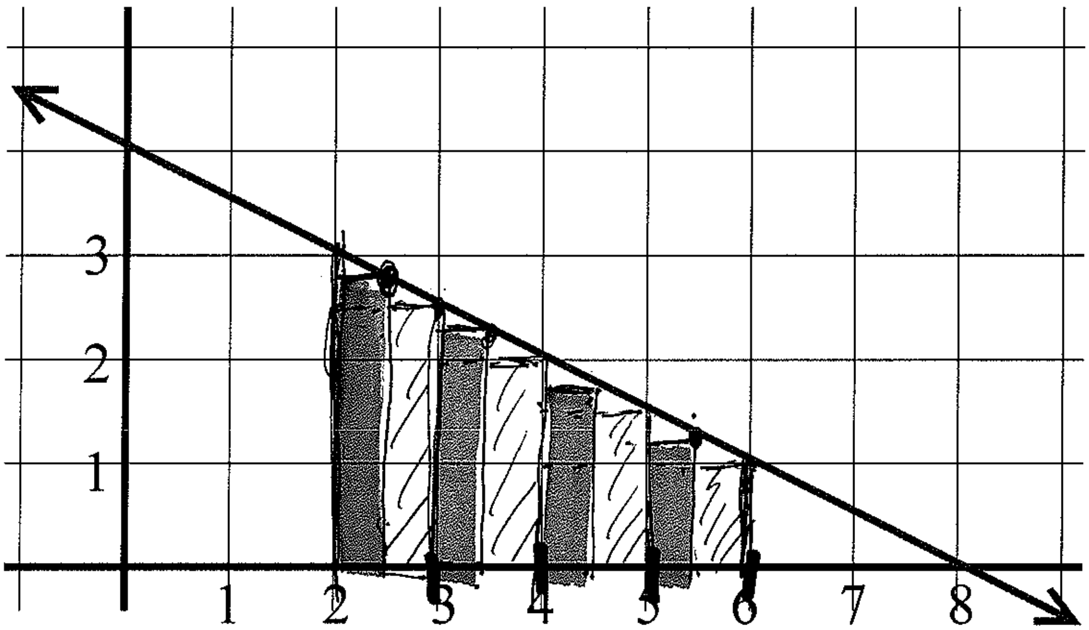
$$= \boxed{\tfrac{5}{2}(1) + 2(1) + \tfrac{3}{2}(1) + 1(1) = 7} < 8$$

since inscribed

width width

height height

Inscribed rectangles with $\Delta x = \frac{1}{2}$:



$$f(\tfrac{5}{2})(\tfrac{1}{2}) + f(3)(\tfrac{1}{2}) + f(\tfrac{7}{2})(\tfrac{1}{2}) + f(4)(\tfrac{1}{2})$$

height width

height width

$$+f(\tfrac{9}{2})(\tfrac{1}{2}) + f(5)(\tfrac{1}{2}) + f(\tfrac{11}{2})(\tfrac{1}{2}) + f(6)(\tfrac{1}{2})$$

$$= [-\tfrac{1}{2}(\tfrac{5}{2}) + 4](\tfrac{1}{2}) + [-\tfrac{1}{2}(3) + 4](\tfrac{1}{2}) + [-\tfrac{1}{2}(\tfrac{7}{2}) + 4](\tfrac{1}{2})$$

$$+ [-\tfrac{1}{2}(4) + 4](\tfrac{1}{2}) + [-\tfrac{1}{2}(\tfrac{9}{2}) + 4](\tfrac{1}{2}) + [-\tfrac{1}{2}(5) + 4](\tfrac{1}{2})$$

$$+ [-\tfrac{1}{2}(\tfrac{11}{2}) + 4](\tfrac{1}{2}) + [-\tfrac{1}{2}(6) + 4](\tfrac{1}{2})$$

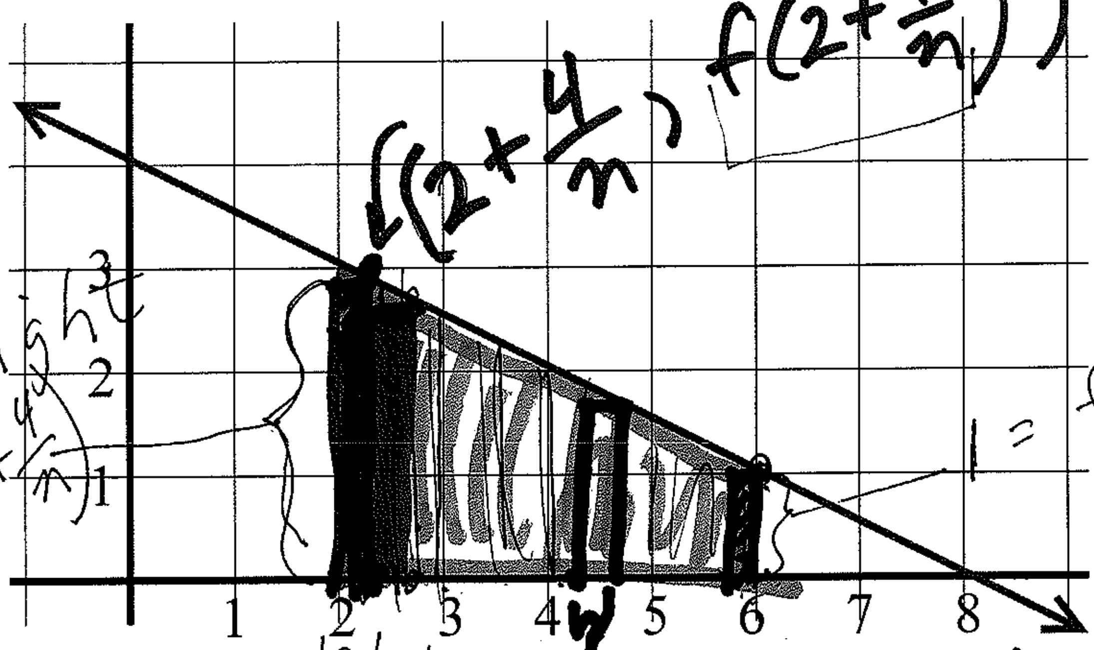
$$= \tfrac{11}{4}(\tfrac{1}{2}) + \tfrac{5}{2}(\tfrac{1}{2}) + \tfrac{9}{4}(\tfrac{1}{2}) + 2(\tfrac{1}{2}) + \tfrac{7}{4}(\tfrac{1}{2}) + \tfrac{3}{2}(\tfrac{1}{2}) + \tfrac{5}{4}(\tfrac{1}{2}) + 1(\tfrac{1}{2})$$

$$= \frac{15}{2}$$

$$7 < \frac{15}{2} < 8$$

change in x = width

Inscribed rectangles with $\Delta x = \frac{6-2}{n} = \frac{4}{n}$:



width

$\frac{4}{n}$

width $= \frac{4}{n}$

height · width $\qquad$ height · width $\qquad$ height width

$f\left(2+\frac{4}{n}\right) \cdot \frac{4}{n} + \left[f\left(2+\frac{4}{n}+\frac{4}{n}\right)\right]\left(\frac{4}{n}\right) + \ldots\ldots f(6)\left(\frac{4}{n}\right)$

Sum of the area of rectangles

= sum of heights × widths $= \sum f(x_i)\left(\frac{4}{n}\right)$

Total area estimate $= \sum f(x_i) \cdot \Delta x$

height · width

over-estimate

Circumscribed rectangles with $\Delta x = 1$: width

$$\sum \underbrace{f(x_i)}_{\text{height}} \underbrace{\Delta x}_{\text{width}} = \sum_{i=2}^{5} f(i)(1) = \sum_{i=1}^{4} f(i+1)(1)$$

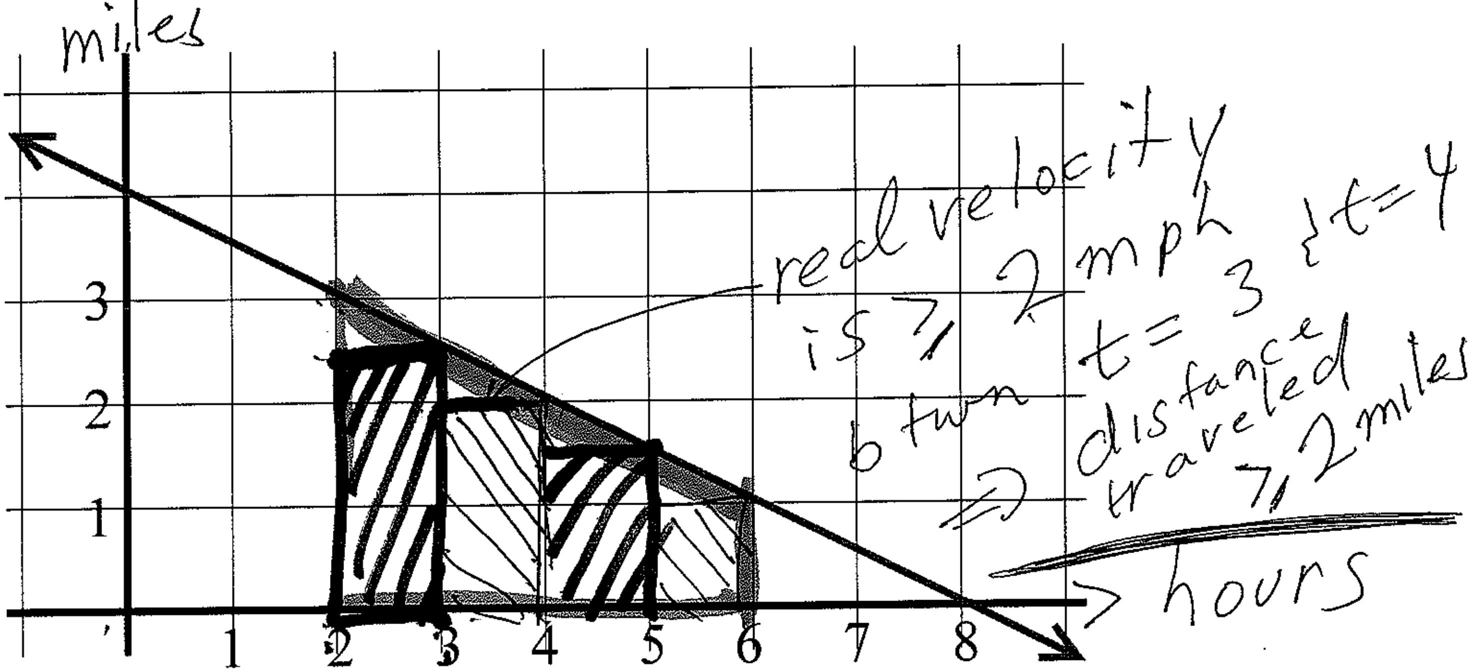
$f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) =$

$$= [-\tfrac{1}{2}(2) + 4](1) + [-\tfrac{1}{2}(3) + 4](1) + [-\tfrac{1}{2}(4) + 4](1) + [-\tfrac{1}{2}(5) + 4](1)$$

$$= 3 + \tfrac{5}{2}(1) + 2(1) + \tfrac{3}{2}(1) = 9 > 8$$

Estimate the distance traveled between $t = 2$ and $t = 6$ if the velocity is given by the function $f(t) = -\frac{1}{2}t + 4. = v(t)$ mph

Estimate using inscribed rectangles with $\Delta t = 1$:



Under estimate of distance traveled = sum of areas of rectangles

$f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) =$

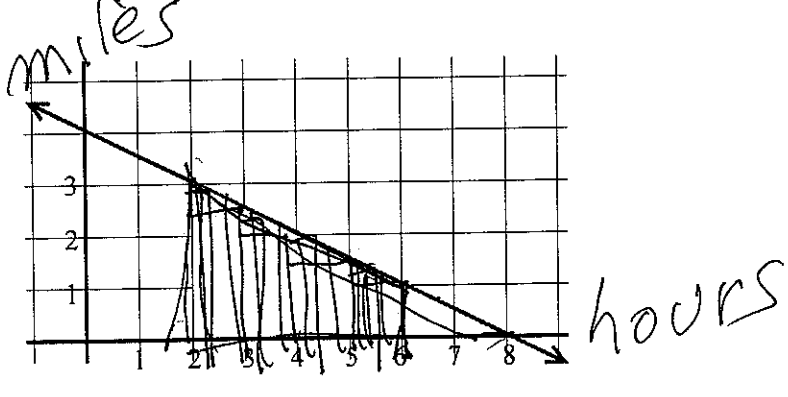
$= [-\frac{1}{2}(3) + 4](1) + [-\frac{1}{2}(4) + 4](1)$
$\quad + [-\frac{1}{2}(5) + 4](1) + [-\frac{1}{2}(6) + 4](1)$

$= \frac{5}{2}(1) + 2(1) + \frac{3}{2}(1) + 1(1) = 7$ miles

v t height width

velocity × time = distance

Find the distance traveled between $t = 2$ and $t = 6$ if the velocity is given by the function $f(t) = -\frac{1}{2}t + 4$.



Method 1: In this case our function is very simple, so we can determine the area without calculus:

$\frac{1}{2}BH - \frac{1}{2}bh = 9 - 1 = 8$ miles

Method 2: Use calculus by estimating with rectangles and taking limit.

$$\text{Area} = = lim_{n\to\infty}\Sigma_{i=1}^{n} \overbrace{f(a + \tfrac{(b-a)i}{n})}^{\text{height}}\overbrace{(\tfrac{b-a}{n})}^{\text{width}}$$
$$= lim_{n\to\infty}\Sigma_{i=1}^{n} f(2 + \tfrac{4i}{n})(\tfrac{4}{n})$$
$$= lim_{n\to\infty}\Sigma_{i=1}^{n} [-\tfrac{1}{2}(2 + \tfrac{4i}{n}) + 4](\tfrac{4}{n}) = 8$$

Method 3 (section 5.3): Use calculus by integrating.

$$\int_2^6 (-\tfrac{1}{2}t + 4)dt = (-\tfrac{1}{4}t^2 + 4t)\Big|_2^6$$
$$= (-\tfrac{1}{4}(6)^2 + 4(6)) - (-\tfrac{1}{4}(2)^2 + 4(2))$$
$$= -9 + 24 - (-1 + 8) = 15 - 7 = 8$$

$f > 0$

Defn: $\int_a^b f(x)dx = lim_{n\to\infty}\Sigma_{i=1}^n \overbrace{f(x_i)}^{\text{height}}\overbrace{\Delta x}^{\text{width}}$ = Area under curve

If f is continuous, can use inscribed rectangles, circumscribed rectangles, all left-hand endpoints, all right-hand endpoints, or all midpoints, etc.

If $\Delta x = \frac{b-a}{n}$ and if right-hand endpoints are used, then $x_i = a + i\Delta x = a + \frac{(b-a)i}{n}$

$$\int_a^b f(x)dx = lim_{n\to\infty}\Sigma_{i=1}^n f(a + \frac{(b-a)i}{n})(\frac{b-a}{n})$$