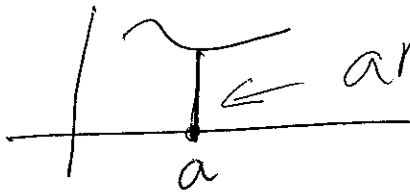
5.4

Properties of the definite integral

$\int_a^a f(x)dx = 0$

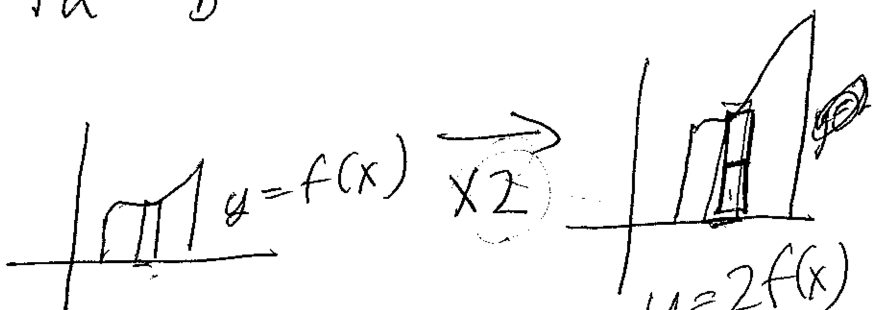


area of line segment $= 0$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int_a^b kf(x)dx = k\int_a^b f(x)dx$

$\underbrace{[k\, f(x_i)]}_{\text{height}}\ \underbrace{dx}_{\text{width}}$

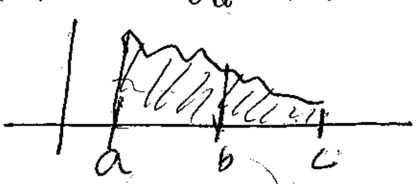


used

$\int_a^b (f_1 + f_2)(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$

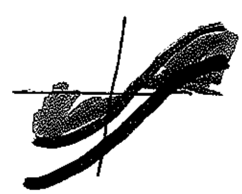
$(f_1(x_i) + f_2(x_i))\,dx$

$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$ ← we will use later today for & have used for piecewise defined fns

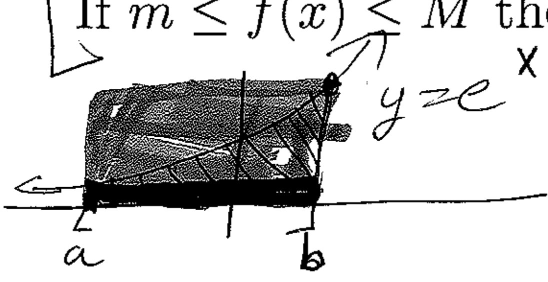


If $f_1(x) \le f_2(x)$, then $\int_a^b f_1(x)dx \le \int_a^b f_2(x)dx$

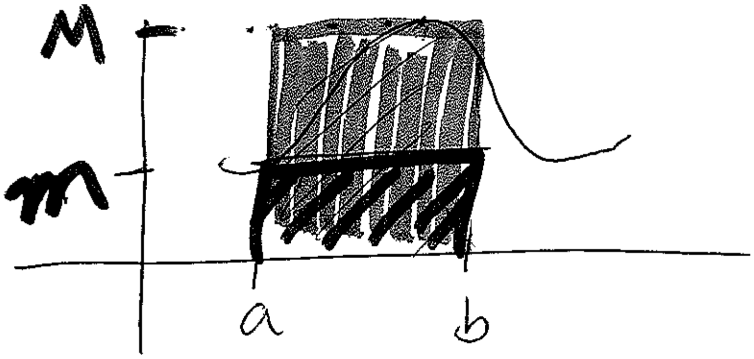
larger height $\Rightarrow$ larger net area



If $m \le f(x) \le M$ then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$



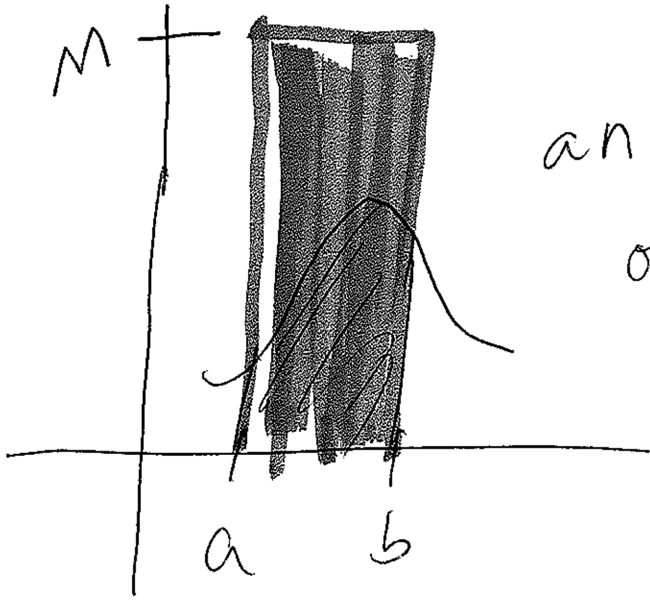
$\underbrace{e^a}_{\text{height}}\underbrace{(b-a)}_{\text{width}} \le \text{area} \le e^b(b-a)$



$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

1 inscribed rectangles

1 circumscribed rectangle

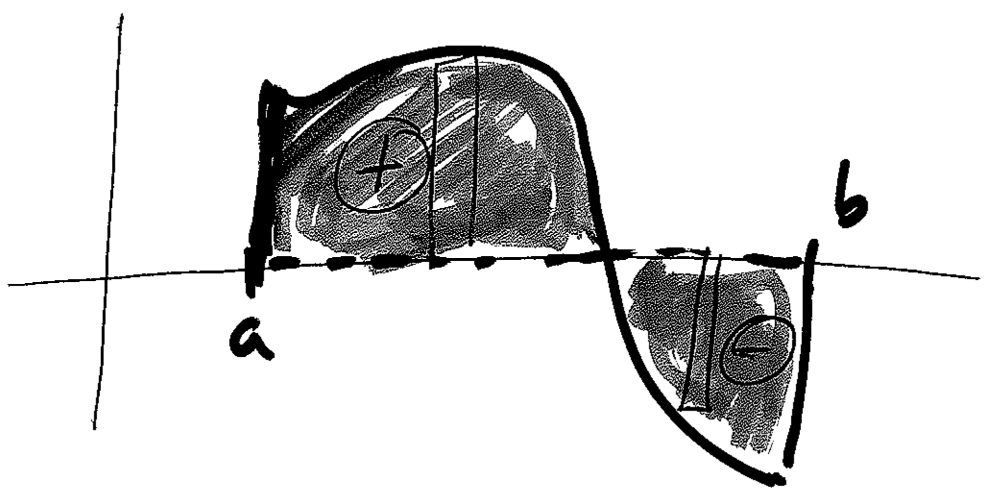


any upper bound on $f(x)$ can be use to bound the area $\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \text{net area btwn } y = f(x) \text{ \& } \begin{bmatrix} x\text{-axis} \\ y = 0 \end{bmatrix}$$

$$x = a \,,\; x = b$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$



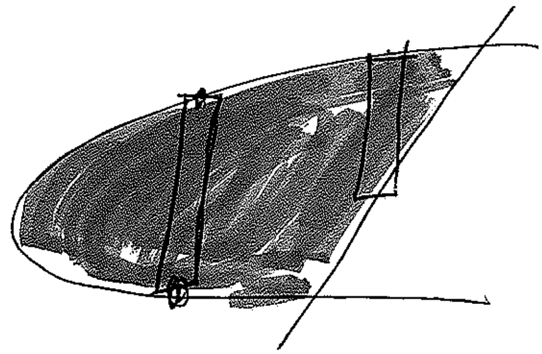
FYI

Find the area between the curve $y^2 = 2x-2$ and $y = x-5$.

Use vertical rectangles:

1.) Find points of intersection between the two curves.

$y^2 = 2x - 2$ and $y = x - 5$.



$(x-5)^2 = 2x - 2$

$x^2 - 10x + 25 = 2x - 2$

$x^2 - 12x + 27 = 0$

$(x-3)(x-9) = 0$. Hence $x = 3, 9$.

2.) Determine which is larger.

Between 1 and 3: $\sqrt{2x-2} > -\sqrt{2x-2}$

Between 3 and 9: $\sqrt{2x-2} > x - 5$

3.) Write as integral(s)

Note that between 1 and 3, the height of the rectangles is $\sqrt{2x-2} - (-\sqrt{2x-2})$ and the width is dx.

Note that between 3 and 9, the height of the rectangles is $\sqrt{2x-2} - (x-5)$ and the width is dx.

$\int_1^3 [\sqrt{2x-2} - (-\sqrt{2x-2})]dx + \int_3^9 [\sqrt{2x-2} - (x-5)]dx$

4.) Evaluate the integral

$\int_1^3 [2\sqrt{2x-2}]dx + \int_3^9 [\sqrt{2x-2} - (x-5)]dx$

$= \int_1^3 [2\sqrt{2x-2}]dx + \int_3^9 (\sqrt{2x-2})dx - \int_3^9 (x-5)dx$

Let $u = 2x - 2$, $du = 2dx$,
$x = 1 : u = 2(1) - 2 = 0;$
$x = 3 : u = 2(3) - 2 = 4;$
$x = 9 : u = 2(9) - 2 = 16$

$= \int_0^4 u^{\frac{1}{2}} du + \int_4^{16} \frac{1}{2} u^{\frac{1}{2}} du + \int_3^9 (-x+5)dx$

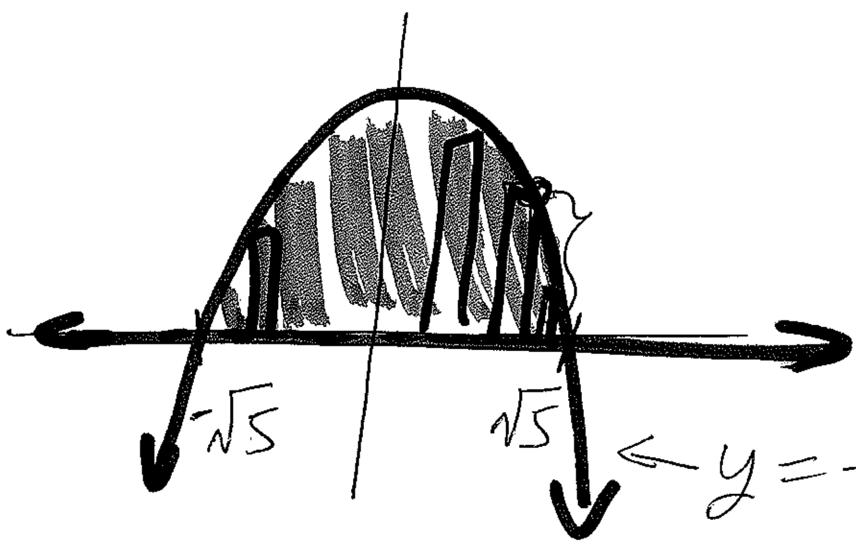
$= \frac{2}{3} u^{\frac{3}{2}} |_0^4 + \frac{1}{3} u^{\frac{3}{2}} |_4^{16} + (-\frac{1}{2}x^2 + 5x)|_3^9$

$$= \frac{2}{3}(4^{\frac{3}{2}} - 0^{\frac{3}{2}}) + \frac{1}{3}(16^{\frac{3}{2}} - 4^{\frac{3}{2}}) + (-\frac{1}{2}(9)^2 + 5(9)) \\ -(-\frac{1}{2}(3)^2 + 5(3))$$

$= \frac{1}{3}[2(8) + 64 - 8] - \frac{81}{2} + 45 + \frac{9}{2} - 15 = 16$

$= \frac{72}{3} - \frac{72}{2} + 30 = 24 - 36 + 30 = 18$

EX: Find area bounded btwn $y = -x^2 + 5$ and $y = 0$ (x-axis)



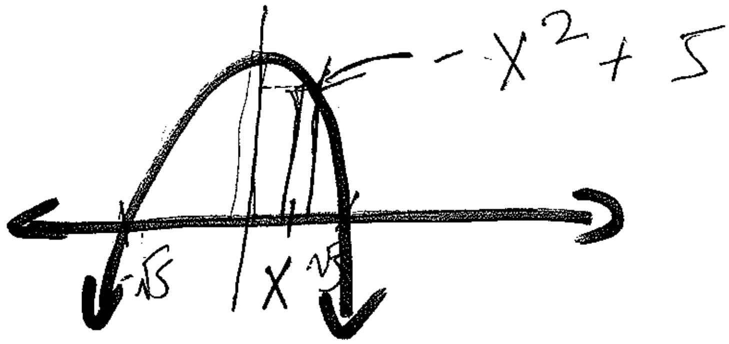
① Find intersection pts
($\Rightarrow$ same y-value

$$0 = -x^2 + 5$$

$$x^2 = 5 \Rightarrow x = \pm\sqrt{5}$$

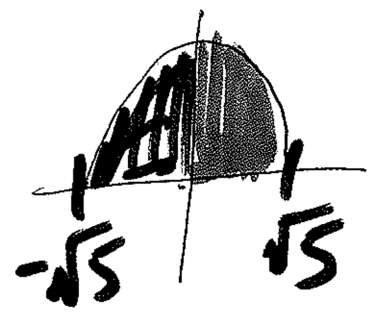
$$\int_{-\sqrt{5}}^{\sqrt{5}} (\text{heights})(\text{widths})$$

dx



$$\int_{-\sqrt{5}}^{\sqrt{5}} (-x^2+5)\,dx$$

$$= -\frac{x^3}{3} + 5x \Big|_{-\sqrt{5}}^{\sqrt{5}}$$



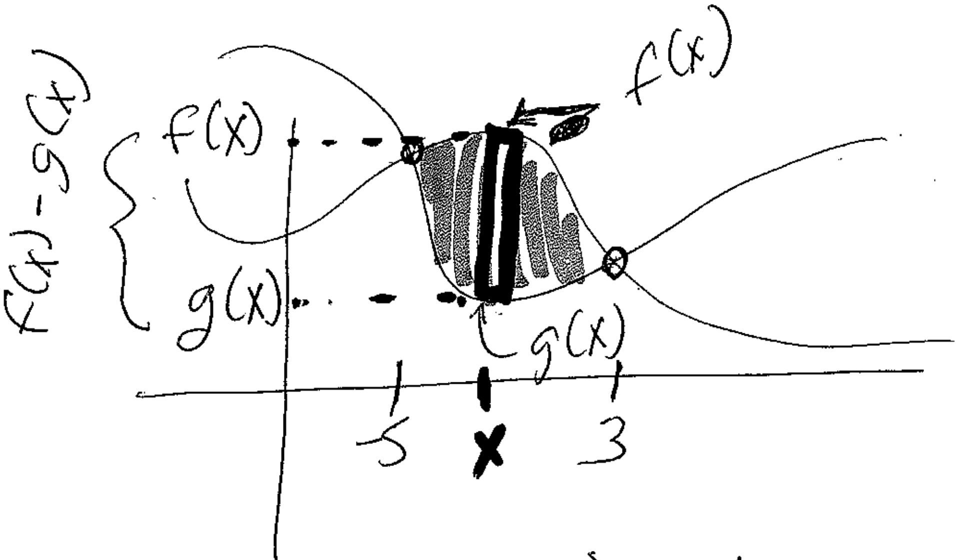
$$\left(-\frac{5\sqrt{5}}{3} + 5\sqrt{5}\right) - \left(\frac{5\sqrt{5}}{3} - 5\sqrt{5}\right)$$

$$= -\frac{10\sqrt{5}}{3} + 10\sqrt{5}$$

$$= \frac{20\sqrt{5}}{3}$$

Find area bounded by

$y = -x^2 + 5$ & $y = 2x - 10$



height = larger fn − smaller fn on $[-5, 3]$

① Find intercepts ($\Rightarrow$ same y-value)

$$-x^2 + 5 = 2x - 10$$

$$0 = x^2 + 2x - 15$$

$$0 = (x+5)(x-3)$$

$$x = -5, 3$$

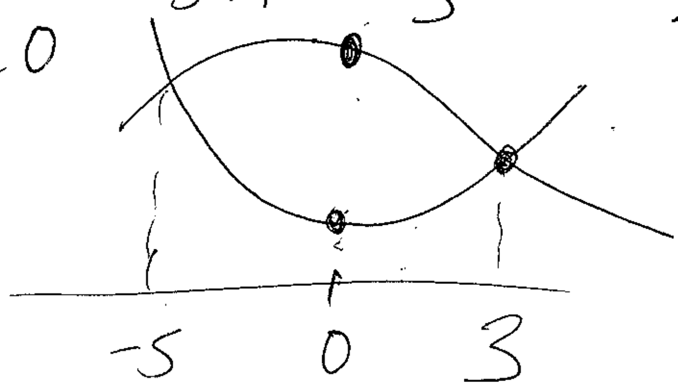
$$\int_{-5}^{3} (\text{heights})\, dx$$

② Determine which fn is larger on $[-5, 3]$

method 1 algebraically

$$\boxed{-x^2+5 > 2x-10}$$

check $x=0$ since $5 > -10$ when $x=0$



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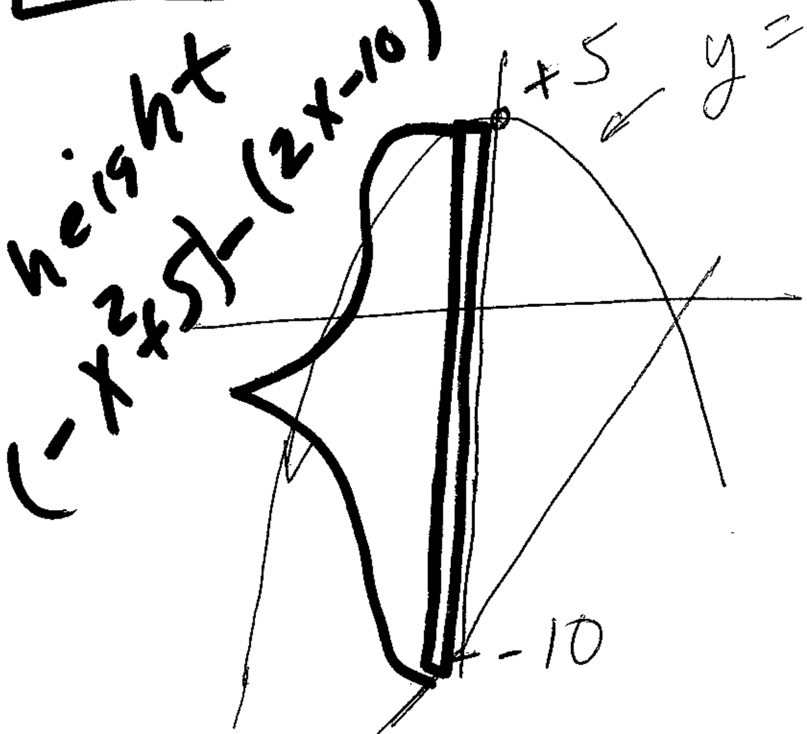
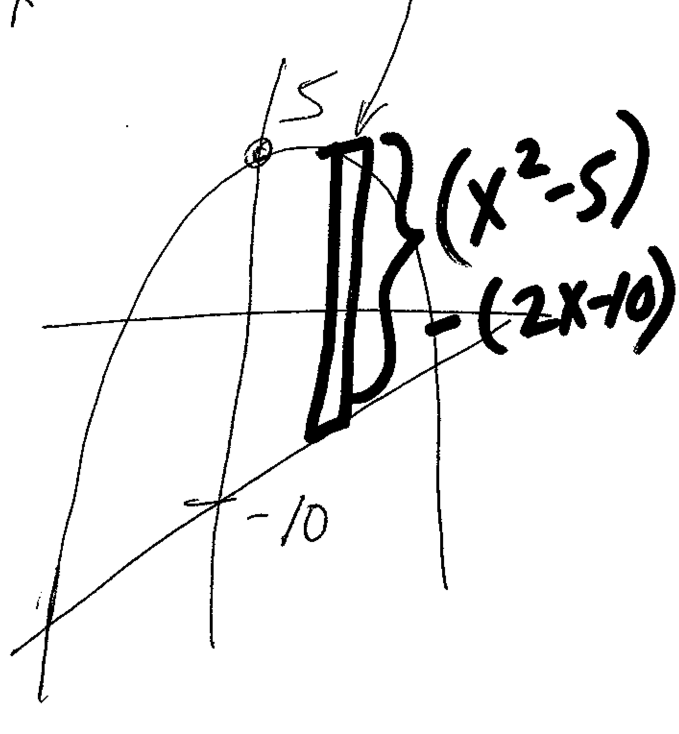
method 2 graphically $y = -x^2+5$ $x^2-5 > 2x-10$





$$-x^2+5 > 2x-10$$

$$\Rightarrow \text{height} = (-x^2+5) - (2x-10)$$

$$\text{Area } \int_{-5}^{3} \underbrace{\left[(-x^2+5) - (2x-10)\right]}_{\text{height}} \underbrace{dx}_{\text{width}}$$

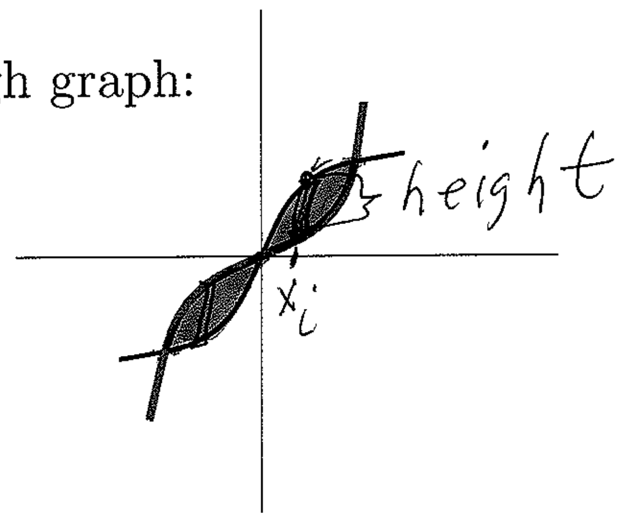
Find the area bounded by the functions $y = 2x^3$ and $y = 2x^{\frac{1}{3}}$.

1.) Find points of intersection:

$2x^3 = 2x^{\frac{1}{3}}$ implies $x^3 = x^{\frac{1}{3}}$ implies $x^9 = x$. Thus $x^9 - x = x(x^8 - 1) = 0$.
Hence $x = 0$ and $x^8 - 1 = 0$. $x^8 = 1$ implies $x = 1, -1$

Hence the functions $y = 2x^3$ and $y = 2x^{\frac{1}{3}}$ intersect when $x = -1, 0, 1$

2.) Draw a rough graph:



3.) Find area:

Use vertical rectangles:

$$\int_{-1}^{0}[2x^3 - 2x^{\frac{1}{3}}]dx + \int_0^1[2x^{\frac{1}{3}} - 2x^3]dx$$

Ex: Find area btwn

$y = 2x^3$ & $y = 2x^{1/3}$

① Find intercepts

($\Rightarrow$ same y-value)

$$\cancel{2}x^3 = \cancel{2}x^{1/3}$$

$$x^3 = x^{1/3}$$

$$x^9 = x$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$x = 0, \quad x^8 - 1 = 0$$

$$x^8 = 1$$

$$x = 0 \quad x = \pm 1 \leftarrow \text{check}$$

$2(\pm 1)^3 = 2(\pm 1)^{1/3}$

$2(0)^3 = 2(0)^{1/3}$ ✓

Alternatively

$$x^3 = x^{1/3}$$

$$x^3 - x^{1/3} = 0$$

$$x^{1/3}(x^{8/3} - 1) = 0$$

$$x^{8/3} = 1 \Rightarrow x = \pm 1$$

$$x = 0$$

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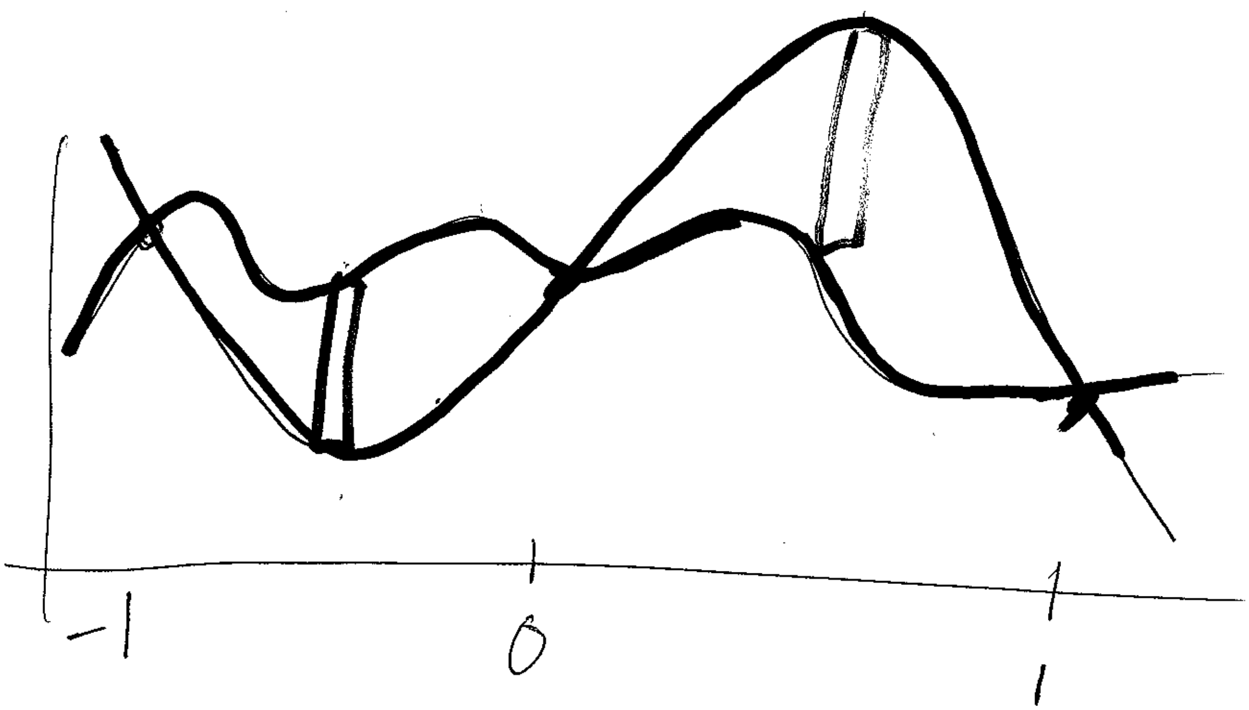
(2) Determine height

Find which fn is larger



Which is larger on $[-1, 0]$

algebraically

$$2x^3 \geq 2x^{1/3}$$

$$2\left(-\frac{1}{8}\right)^3 > 2\left(-\frac{1}{8}\right)^{1/3} = -1$$

Which fn is large on $[0,1]$

$$2\left(+\frac{1}{8}\right)^3 < 2\left(+\frac{1}{8}\right)^{1/3} = 1$$

$$2x^3 < 2x^{1/3}$$

$$\int_{-1}^{0}\left[2x^3 - 2x^{1/3}\right]dx + \int_0^1\left[2x^{1/3} - 2x^3\right]dx$$
$$= \text{Area}$$