Defn: M is an n-dimensional manifold (with boundary) if

1.) For all $x \in M$, there exists a neighborhood V_x such that V_x is homeomorphic to an open set in R^n or ${R_+}^n$

2.) M is T_2 and ...

Give an example of a topological space which satisfies (1), but is not T_2.

Answer: Friday's Lecture.

M is a closed manifold if M is a compact manifold without boundary.

Wild knot:

Alexander horned sphere: see handout

To avoid such pathologies, we will work in the differentiable (C^∞) or piecewise linear (PL) category.

Examples of n-manifolds:

$D^n = B^n = \{\mathbf{x} \in R^n \mid ||x|| \leq 1\}$

$S^n = \{\mathbf{x} \in R^{n+1} \mid ||x|| = 1\} = \partial B^{n+1}$

$P^n = S^n/(\mathbf{x} \sim -\mathbf{x})$

$T^n = S^1 \times S^1 \times ... \times S^1$

Forming new manifolds from old manifolds:

If M is an m-manifold and N is an n-manifold, then $M \times N$ is a (m+n)-manifold.

If M is an m-manifold, then ∂M is an (m-1)-manifold.

Suppose M and N are n-manifolds and
f : a component of $\partial M \rightarrow$ a component of ∂N is a homeomorphism, then

$$M \cup_f N = M \cup N/(x \sim f(x))$$

In particular, $M \# N = (M - B^n) \cup_i (N - B^n)$ where $i : S^{n-1} \rightarrow S^{n-1}$.

$F_g = \#T^2 = (S^2 - \cup_{i=1}^{2g} D^2) \cup (\cup_{i=1}^{g} A^2)$ where $A^2 =$ annulus

$N_g = \#P^2 = (S^2 - \cup_{i=1}^{g} D^2) \cup (\cup_{i=1}^{g} V^2)$ where $V^2 =$ mobius band

Euler characteristic $= \chi(M) =$ vertices - edges + faces - ...
$= \Sigma_{i=0}^{\infty}(-1)^i \alpha_i(M)$ where $\alpha_i(M) =$ number of i cells.
$= \Sigma_{i=0}^{\infty}(-1)^i \beta_i(M)$ where $\beta_i(M) = dim H_i(M)$

$\chi(M_1 \cup_F M_2) = \chi(M_1) + \chi(M_2) - \chi(F)$

$\chi(S^{2n-1}) = 0.$ $\chi(S^{2n}) = 2.$ $\chi(D^n) = 1.$

If S is a surface (compact connected 2-manifold) consisting of disjoint disks with bands attached, then

$$\chi(S) = \# \text{ of disks - } \# \text{ of bands.}$$

$\chi(T^2) = 0.$ $\chi(T^2 \# T^2) = -2,$ $\chi(F_g) = 2 - 2g$
$\chi(P^2) = 1.$ $\chi(P^2 \# P^2) = 0,$ $\chi(N_g) = 2 - g$

Casson: "For three-dimensional topology, intuitive understanding is much more important than technical details."