

Thm 5.3: Suppose  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2 \in C^1$

$$T(u, v) = (T_1(u, v), T_2(u, v)) = (x(u, v), y(u, v))$$

If  $D^* \subset \mathbf{R}^2$  and if  $f : T(D^*) \rightarrow \mathbf{R}$  is integrable, then

$$\int \int_{T(D^*)} f(x, y) dx dy =$$

$$\int \int_{D^*} f((x(u, v), y(u, v))) \text{abs}(\det(DT(u, v))) du dv$$

Ex: Evaluate  $\int \int_{T(D^*)} xy dx dy$

where  $T(D^*) =$  the parallelogram where the vectors  $(4, 1)$  and  $(2, 3)$  form two of its sides.

Find  $D^*$  and  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  such that  $T(D^*)$  is the parallelogram described above.

Note  $T$  is a linear transformation.

$$T : \mathbf{R}^2 \rightarrow \mathbf{R}^2, T(u, v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$T(1, 0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$T(0, 1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ,

$$T(u, v) = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4u + 2v \\ u + 3v \end{pmatrix} = \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$$

Let  $x = 4u + 2v$  and  $y = u + 3v$

$$DT = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \text{ and } \text{abs}(\det \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}) = 12 - 2 = 10$$

$$\begin{aligned} & \int \int_{T(D^*)} (x + y) dx dy \\ &= \int \int_{D^*} f((x(u, v), y(u, v))) \text{abs}(\det(DT(u, v))) du dv \\ &= \int \int_{D^*} (4u + 2v)(u + 3v)(10) du dv \\ &= 10 \int_0^1 \int_0^1 (4u^2 + 14uv + 6v^2) du dv \\ &= 10 \int_0^1 \left( \frac{4}{3}u^3 + 7u^2v + 6uv^2 \Big|_0^1 \right) dv \\ &= 10 \int_0^1 \left( \frac{4}{3} + 7v + 6v^2 \right) dv \\ &= 10 \left( \frac{4}{3}v + \frac{7}{2}v^2 + 2v^3 \Big|_0^1 \right) \\ &= 10 \left( \frac{4}{3} + \frac{7}{2} + 2 \right) \end{aligned}$$