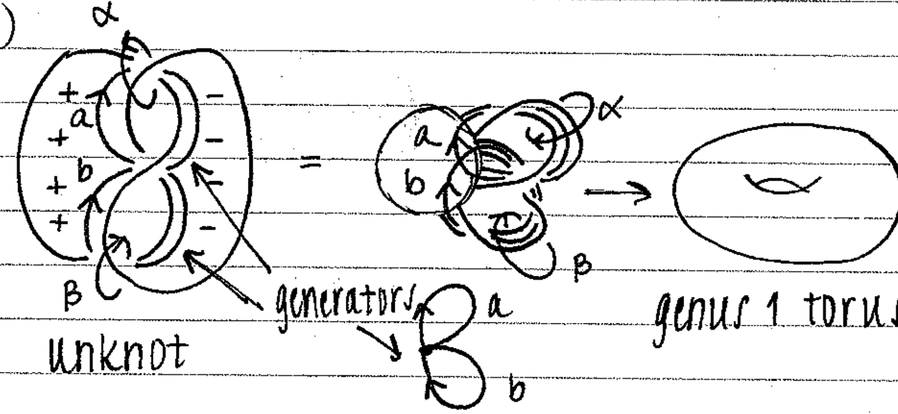
Thursday, February 19, 2010

Chapter 6 : $H_1(X_K)$

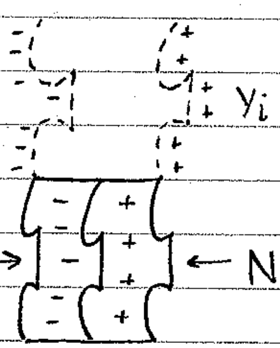
Note: Rolfsen does similar example but with trefoil, not unknot.

$M =$ 

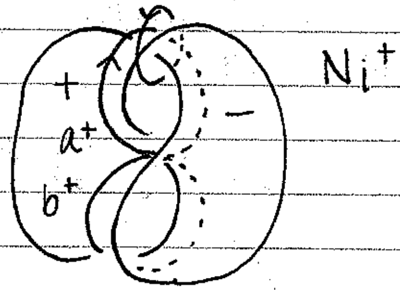

(Seifert surface)

$H_1(M) = \langle a, b \mid a+b = b+a \rangle = (a, b)$ (denotes free abelian group generated by a & b)

$H_1(\underbrace{S^3 \setminus M}_{Y_i}) = (\alpha, \beta)$

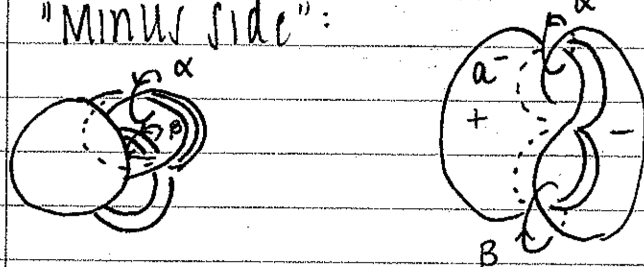



"Plus side":




$a^+ = -\alpha$ (opp. orient.)
$b^+ = \alpha$

"Minus side":




$a^- = \beta - \alpha$ (tough to see!)
$b^- = 0$ (pulls right off, no twists!)

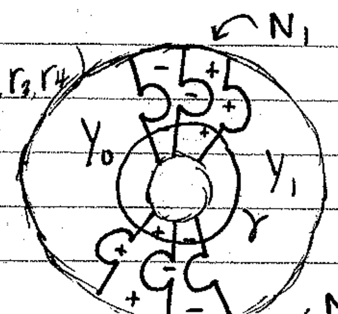
<u>Claim</u>: $H_1(\tilde{X}_2) = (\alpha_0, \beta_0, \alpha, \beta, \gamma \mid r_1, r_2, r_3, r_4)$

$r_1: a_0^- = a_0^+ \Rightarrow \beta_1 - \alpha_1 = -\alpha_0$

$r_2: b_0^- = b_0^+ \Rightarrow 0 = \alpha_0$

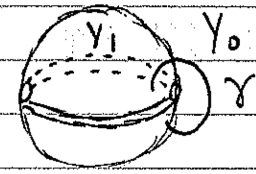
$r_3: a_1^+ = a_1^- \Rightarrow \beta_0 - \alpha_0 = -\alpha_1$

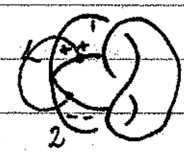
$r_4: b_1^- = b_1^+ \Rightarrow 0 = \alpha_1$



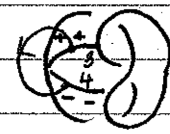
"Better picture for $\gamma$"

thickened s.s. of disk



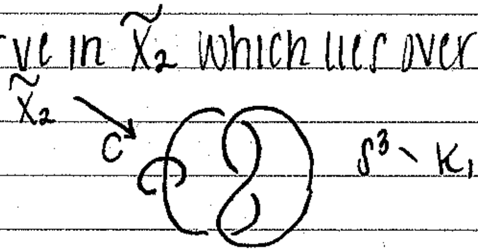




$Y_0$



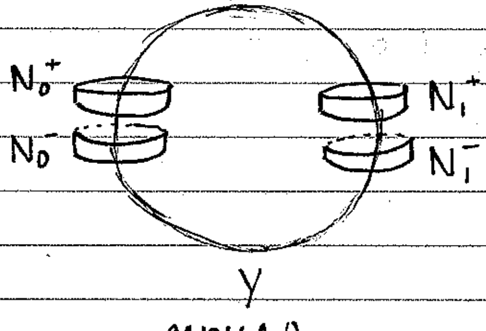

$Y_1$

Let $\Gamma$ be a curve in $\tilde{X}_2$ which lies over $C$ where $lk(C, K) = 1$

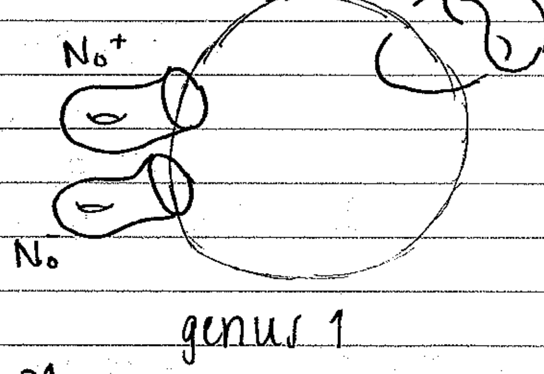



Let $Y' = Y_0 \cup Y_1 \cup \Gamma$
$N' = N_0 \cup N_1 \cup \Gamma$
$Y' \cup N' = \tilde{X}_2$ (double cover)
$Y' \cap N' = N_0^+ \cup N_0^- \cup N_1^+ \cup N_1^- \cup \Gamma$
} all connected by $\Gamma$

$H_1(N' \cap Y') = (\gamma, a_0^+, b_0^+, a_0^-, b_0^-, a_1^+, b_1^+, a_1^-, b_1^-)$

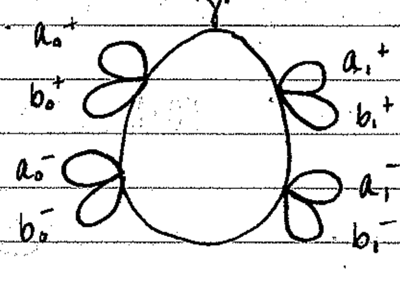



genus 0




genus 1

Deformation retracts to:



$H_1(N') = (\gamma', a_0, b_0, a_1, b_1)$

$H_1(Y') = (\gamma'', \alpha_0, \beta_0, \alpha_1, \beta_1)$ (computed via Mayer-Vietoris)

Mayer-Vietoris:

$$\to H_1(N'\cap Y') \xrightarrow{f} H_1(N') \oplus H_1(Y') \xrightarrow{g} H_1(\tilde{X}_2) \to \overset{\tilde{H}_0(N'\cap Y')}{\overset{\|}{0}}$$

$$H_1(\tilde{X}_2) = \frac{H_1(N') \oplus H_1(Y')}{\ker g = \operatorname{im} f}$$

$$\begin{aligned}
H_1(N'\cap Y') &\xrightarrow{f} H_1(N') \oplus H_1(Y') \\
x &\longmapsto (x, x) \\
\gamma &\longmapsto (\gamma', \gamma'') \\
a_0^+ &\longmapsto (a_0, -\alpha_0) \\
b_0^+ &\longmapsto (b_0, \alpha_0) \\
a_0^- &\longmapsto (a_0, \beta_1 - \alpha_1) \\
b_0^- &\longmapsto (b_0, 0) \\
a_1^+ &\longmapsto (a_1, -\alpha_1) \\
b_1^+ &\longmapsto (b_1, \alpha_1) \\
a_1^- &\longmapsto (a_1, \beta_0 - \alpha_0) \\
b_1^- &\longmapsto (b_1, 0)
\end{aligned}$$

Thus, $H_1(\tilde{X}_2) = (\gamma, \alpha_0, \beta_0, \alpha_1, \beta_1 \mid -\alpha_0 = \beta_1 - \alpha_1,\ \alpha_0 = 0,\ -\alpha_1 = \beta_0 - \alpha_0,\ \alpha_1 = 0) = \mathbb{Z} \oplus 0 = \mathbb{Z}$

Generalizing:

k-fold cover

$H_1(X_k) = (\gamma, \alpha_i, \beta_i;\ i = 0, \ldots, k-1 \mid k \text{ relations}) = \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_n}$

If M has genus g & $\partial M$ is a 1-comp. knot