3.8: Mechanical and Electrical Vibrations

Trig background:

$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$Acos(\omega_0 t) + Bsin(\omega_0 t)$
$= Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t)$
$= Rcos(\omega_0 t - \delta)$

Amplitude $= R$
frequency $= \omega_0$ (measured in radians per unit time).
period $= \frac{2\pi}{\omega_0}$
phase (displacement) $= \delta$

Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$
$$mg - kL = 0, \qquad F_{viscous}(t) = \gamma u'(t)$$

m= mass,
k = spring force proportionality constant,
γ = damping force proportionality constant
g = 9.8 m/sec

Electrical Vibrations:

$$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

L = inductance (henrys),
R = resistance (ohms)
C = capacitance (farads)
$Q(t)$ = charge at time t (coulombs)
$I(t)$ = current at time t (amperes)
$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad = 1 henry · 1 amperes/ 1 second

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$

$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t} + \psi(t)$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t} + \psi(t)$

$$\gamma^2 - 4km < 0: \; u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t) + \psi(t)$$
$$= e^{-\frac{\gamma t}{2m}} Rcos(\mu t - \delta) + \psi(t)$$
where $A = Rcos(\delta)$, $B = Rsin(\delta)$

$\mu =$ quasi frequency, $\frac{2\pi}{\mu} =$ quasi period

Note if $\gamma = 0$, then

Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

[$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
Hence $u(t) = Acos\mu t + Bsin\mu t$ since $\gamma = 0$].

$2u''(t) + 16u(t) = 0$

$u''(t) + 8u(t) = 0$

$u(0) = 1$, $u'(0) = -\sqrt{8}$

$r^2 + 8 = 0 \rightarrow r^2 = -8 \rightarrow r = \sqrt{-8} = i\sqrt{8} = 0 + i\sqrt{8}$