

The median for this midterm was 72.25. This isn't bad considering how many of you did not know definitions. I expect grades to improve on the final exam if you learn definitions (make flashcards) and follow (proof) writing advice. Note I do take improvement into consideration as discussed in class and on ICON.

It is easy to make (false) assumptions. What you learn in this and other math classes will hopefully help you make fewer assumptions and to be more precise. The final exam will have a similar format to this midterm, so that you can do extra problems in case you make an incorrect assumption in other problem(s).

Description of grading scheme and advice for final exam (and HW, quizzes, etc).

For problem #2, I did allow many/most of you to get away with very poor writing. You should keep in mind that other professors as well as the real world may grade you more harshly. Poor writing can result in costly misunderstandings, so working on improving your writing skills (both English and Math) is highly recommended. But for this class, we will try to grade your English gently (but if we can't infer what you mean, we can't give you points).

For definitions, you must be complete and precise. You can use English and/or math notation, but your definition must be accurate. Thus I recommend flash cards for definitions. As you go through the flash cards, think about examples as well as various parts of the definition and why the parts are included in the definition.

If you don't know definitions, you will likely make mistakes in proofs and in modeling. Knowing definitions gives you more tools to model real life problems.

Note for problems 3 and 4, knowing definitions and theorems can be very helpful when creating examples. For problem 3, some of you gave an example of two non-isomorphic graphs that satisfied your incorrect definition of isomorphic (if two graphs have the same degree sequence, then there exists a bijection between the vertices and between the edges of the two graphs). *incomplete*

For problem 5, note I gave 2 proofs. The second proof uses more notations for describing a walk and thus might be easier to write/read.

I also gave 2 proofs for problem 6, the induction proof. Note you could earn more than half the points, by proving a base case, stating the induction hypothesis, and starting the proof that  $S(m)$  implies  $S(m+1)$  for proof 1 or  $S(\leq m)$  implies  $S(m+1)$  for proof 2. Note that a graph needs to be connected in order to apply the induction hypothesis. For more on induction proofs, please see

[http://homepage.divms.uiowa.edu/idarcy/COURSES/4060/induction\\_cycle.pdf](http://homepage.divms.uiowa.edu/idarcy/COURSES/4060/induction_cycle.pdf)

Note the last problem, #7, was the "easiest" as it followed from definitions (and if you forgot the precise definition for  $k$ -vertex colorable, you could modify the definition of  $k$ -edge colorable).

2.) Describe how to mathematically model the following problem.

The following table shows classes taken by 5 students where an  $x$  in row  $i$ , column  $j$  means student  $i$  is taking class  $j$ . What is the minimum number of time slots for final exams (i.e., final exam times), so that no student has 2 final exams scheduled for the same time.

	Bio	Calc	Java	Physics
Claus		x		x
Irma			x	x
Javier	x			x
Mariel	x		x	
Yuanan	x	x		

(a) Draw the graph that can be used to solve this problem. What do your vertices represent? What do your edges represent?

Vertices represent courses. Edges represent a conflict for a final exam time or equivalently an edge is drawn between two vertices if there is at least one student taking the two courses represented by the two vertices.



(b) State and solve the relevant graph theory problem.

Find  $\chi(G)$ , the chromatic number of  $G$ .

Since  $G$  contains an odd cycle,  $\chi(G) > 2$ . By the coloring below,  $\chi(G) \leq 3$ . Thus  $\chi(G) = 3$

(c) Explain how the solution found in part (b) can be applied to the problem you are modeling.

Each color corresponds to a time slot. Since  $\chi(G) = 3$ , 3 time slots are needed for the final exams so that no student has 2 final exams scheduled for the same time.

4.) Define: The chromatic number,  $\chi(G) = \min\{k \mid G \text{ is } k\text{-vertex colorable}\}$ .

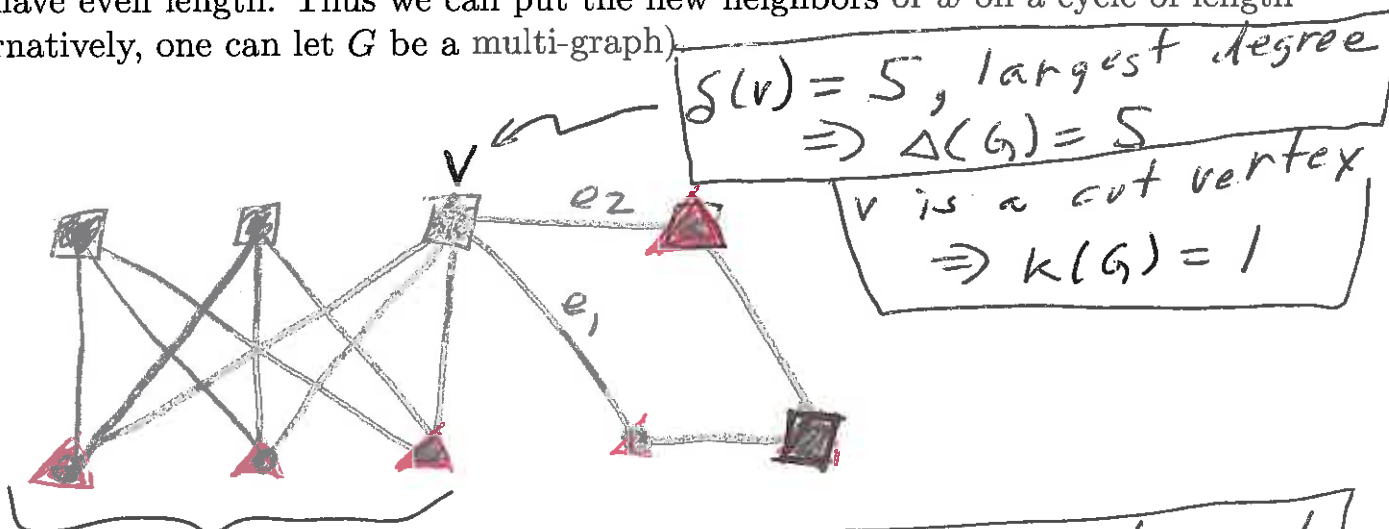
Give an example of a non-planar graph,  $G$ , with  $\kappa(G) = 1$ ,  $\lambda(G) = 2$ ,  $\Delta(G) = 5$ , and  $\chi(G) = 2$ . Justify your answer. Note if your example does not satisfy all the requested conditions, please state which conditions are missing for partial credit.

$\chi(G) = 2$  implies  $G$  is bipartite. We want a bipartite non-planar graph. Note  $K_{3,3}$  is a non-planar and bipartite (alternatively, could try a subdivision of  $K_5$  or  $K_{3,3}$ ).

If we start with  $K_{3,3}$  we can add two new neighbors to one of the vertices,  $w$  to create a graph with  $\Delta(G) = 5$ .

$\kappa(K_{3,3}) = 3$  and  $\lambda(K_{3,3}) = 3$ , so will try to make  $w$  a cut vertex of  $G$ .



Since  $\lambda(G) = 2$ , every pair of vertices should lie on a cycle. Since  $G$  is bipartite, all cycles have even length. Thus we can put the new neighbors of  $w$  on a cycle of length 4 (alternatively, one can let  $G$  be a multi-graph).



$\delta(v) = 5$ , largest degree  
 $\Rightarrow \Delta(G) = 5$   
 $v$  is a cut vertex  
 $\Rightarrow \kappa(G) = 1$

contains  $K_{3,3}$   
 Thus non planar

$\{e_1, e_2\}$  is a vertex cut  
 $\Rightarrow \lambda(G) \leq 2$   
~~Every edge in cycle~~  
 Every edge in cycle  
 $\Rightarrow \lambda(G) \geq 2$   
 Thus  $\lambda(G) = 2$

coloring of vertices  
 or   
 $\Rightarrow \chi(G) = 2$

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6.) Define:  $T$  is a subgraph of  $G$  if

$V(T) \subset V(G)$ ,  $E(T) \subset E(G)$ , and  $\{v \mid \exists \langle u, v \rangle \in E(T)\} \subset V(T)$  (i.e., the endpoints of edges in  $T$  are vertices in  $T$ ).

Let  $G$  be a simple connected graph. Use induction on  $|E(G)|$  to prove that  $G$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G)$ .

**Proof by induction on  $|E(G)|$ :**

Base case:

Suppose  $|E(G)| = 0$ . Since  $G$  is a connected graph with no edges,  $G = (\{v\}, \emptyset) = K_1$ .  
Let  $T = G$ .

Induction hypothesis: <sup>hyp</sup> Suppose that if  $G$  is a simple connected graph with  $|E(G)| = k$ , then  $G$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G)$ . <sub>concl</sub>

Let  $G$  be a simple connected graph with  $|E(G)| = k + 1$ .

Claim:  $G$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G)$ .

Case 1: If  $G$  is a tree, then let  $T = G$ . Then  $V(T) = V(G)$

Case 2:  $G$  is not a tree. Thus  $G$  contains a cycle. Let  $e$  be an edge in this cycle. Then  $e$  is not a cut-edge (bridge).

Thus  $G' = G - e$  is a simple connected graph with  $k$  edges. Thus by the induction hypothesis,  $G'$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G')$ .

Since  $G' = G - e$ ,  $V(G) = V(G') = V(T)$ . Thus  $T$  is a subgraph of  $G$  where  $T$  is a tree and  $V(T) = V(G)$ . Thus the claim holds.

**Alternative induction proof.**

**Proof by induction on  $|E(G)|$ :**

Base case:

Suppose  $|E(G)| = 0$ . Since  $G$  is a connected graph with no edges,  $G = (\{v\}, \emptyset) = K_1$ .  
Let  $T = G$ .

Induction hypothesis: Suppose that if  $G$  is a simple connected graph with  $|E(G)| < k + 1$ , then  $G$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G)$ .

Let  $G$  be a simple connected graph with  $|E(G)| = k + 1$ .

Claim:  $G$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G)$ .

Let  $e \in E(G)$  and let  $G' = G - e$ . Note  $G' = G - e$  is a simple graph with  $k$  edges.

Case 1:  $G'$  is connected.

Thus  $G' = G - e$  is a simple connected graph with  $k$  edges. Thus by the induction hypothesis,  $G'$  contains a subgraph  $T$  where  $T$  is a tree and  $V(T) = V(G')$ .

Since  $G' = G - e$ ,  $V(G) = V(G') = V(T)$ . Thus  $T$  is a subgraph of  $G$  where  $T$  is a tree and  $V(T) = V(G)$ . Thus the claim holds.

Case 2:  $G'$  is not connected. Then  $G'$  has 2 connected components,  $G_1$  and  $G_2$ .

$|E(G_i)| < E(G') = k$ . Thus by the induction hypothesis,  $G_i$  contains a subgraph  $T_i$  where  $T_i$  is a tree and  $V(T_i) = V(G_i)$ .

Let  $T = (V(T_1) \cup V(T_2), E(T_1) \cup E(T_2) \cup \{e\})$  - i.e, create  $T$  from  $T_1$  and  $T_2$  by adding in the edge  $e$ .

Note  $V(T) = V(T_1) \cup V(T_2) = V(G') = V(G)$ . Note  $T$  is connected since  $T_1$  and  $T_2$  are connected and all vertices in  $T_1$  are reachable from all vertices in  $T_2$  via a path that goes through  $e$ .  $T$  is also a tree since  $T_1$  and  $T_2$  are trees and  $e$  is a bridge. Thus  $T$  is a subgraph of  $G$  where  $T$  is a tree and  $V(T) = V(G)$ . Thus the claim holds.