

Theorem 2.2 (Havel-Hakimi): Consider a list $s = [d_1, d_2, \dots, d_n]$ of n numbers in descending (non-increasing) order. This list is graphic if and only if $s^* = [d_1^*, d_2^*, \dots, d_{n-1}^*]$ of $n - 1$ numbers is graphic as well, where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for } i = 1, 2, \dots, d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

I.e., s is graphic iff s^* is graphic.

Proof: Let $s = [d_1, d_2, \dots, d_n]$ where $d_i \geq d_{i+1}$, $d_i \in \{0, 1, 2, \dots\}$

Let $s^* = [d_1^*, d_2^*, \dots, d_{n-1}^*]$ where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for } i = 1, 2, \dots, d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

$$\text{Thus } d_i = \begin{cases} d_1 & i = 1 \\ d_{i-1}^* + 1 & \text{for } i = 2, 3, \dots, d_1 + 1 \\ d_{i-1}^* & \text{otherwise} \end{cases}$$

(\Leftarrow) Suppose s^* is graphic.

Claim s is graphic.

s^* graphic implies \exists simple graph G^* with degree sequence s^* .

Let $G^* = (V^*, E^*)$ where $V^* = \{v_1, \dots, v_{n-1}\}$ and $\delta(v_i) = d_i^*$.

Let $V = \{u, v_1, \dots, v_{n-1}\}$

Let $E = E^* \cup \{< u, v_i > \mid i = 1, \dots, d_1\}$

$$\text{Then } \delta(w) = \begin{cases} d_1 & w = u \\ d_i^* + 1 & w = v_i, i = 1, \dots, d_1 \\ d_i^* & \text{else} \end{cases}$$

Thus $G = (V, E)$ has degree sequence s . Note G is a simple graph. Thus s is graphic.

(\Rightarrow) Suppose s is graphic.

Claim s^* is graphic.

s graphic implies \exists simple graph $G = (V, E)$ with degree sequence s , where $V = \{v_1, \dots, v_n\}$ where $\delta(v_i) = d_i$.

Let $G^* = G - v_1 = (V^*, E^*)$ where $V^* = \{v_2, \dots, v_n\}$ and $E^* = \{e \in E \mid e \text{ is not adjacent to } v_i\}$ ■

Note G^* is a simple graph.

Case 1: Suppose v_1 is adjacent to v_2, \dots, v_{d_1+1}

Removing v_1 reduces the degree of v_2, \dots, v_{d_1+1} by one. Thus G^* has degree sequence s^* . Thus s^* is graphic if case 1 holds.

Case 2: Suppose $\exists j \in \{2, \dots, v_{d_1+1}\}$ such that v_1 is NOT adjacent to v_j .

Then since $\delta(v_1) = d_1$, $\exists \ell > d_1 + 1$ such that v_1 is adjacent to v_ℓ .

See chalkboard for rest of proof.

More elegant proof of s is graphic implies s^* is graphic.

(\Rightarrow) Suppose s is graphic.

Claim s^* is graphic.

s graphic implies \exists simple graph $G = (V, E)$ with degree sequence s , where $V = \{v_1, \dots, v_n\}$ where $\delta(v_i) = d_i$.

Let $N(v_1) = \{w \in V \mid w > v_1, w \in E\}$

Let $A_G = N(v_1) \cap \{v_2, \dots, v_{d_1+1}\}$

Note $0 \leq |A_G| \leq d_1$.

Among all simple graphs G with degree sequence s , choose one such that $|A_G|$ is as large as possible.

Let $G^* = G - v_1 = (V^*, E^*)$ where $V^* = \{v_2, \dots, v_n\}$ and $E^* = \{e \in E \mid e \text{ is not adjacent to } v_i\}$ ■

Note G^* is a simple graph.

Case 1: Suppose $|A_G| = d_1$. Then v_1 is adjacent to v_2, \dots, v_{d_1+1}

Then G^* has degree sequence s^* . Thus s^* is graphic. ■

Case 2: Suppose $|A_G| < d_1$.

Then $\exists v_j, j \in \{2, \dots, d_1 + 1\}$ such that v_1 is not adjacent to v_j in G and $\exists v_\ell, \ell > d_1 + 1$ such that v_1 is adjacent to v_ℓ in G

Note $j \leq d_1 + 1 < \ell$. Thus $\delta(v_j) \geq \delta(v_\ell)$.

Case 2a: Suppose $\delta(v_j) = \delta(v_\ell)$.

Then relabel the vertices v_j and v_ℓ . But by relabeling, we increase $|A_G|$ which contradicts maximality, so case 2a cannot occur.

Case 2b: Suppose $\delta(v_j) > \delta(v_\ell)$.

Then $\exists x \in V$ such that $\langle v_j, x \rangle \in E(G)$, but $\langle v_\ell, x \rangle \notin E(G^*)$

Let $G' = (V, E')$ where $V(G') = V(G)$ and

$$E' = [E(G) - \{\langle v_j, x \rangle, \langle v_\ell, v_1 \rangle\}] \cup \{\langle v_j, v_1 \rangle, \langle v_\ell, x \rangle\}$$

Note G' is a simple graph with degree sequence s . But $|A_{G'}| = |A_G| + 1$, contradicting maximality, so case 2b cannot occur.

Thus case 1 holds and s^* is graphic.