1.7: Linear Independence.

Defn: The set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ is **linearly independent** if and only if the equation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} = \mathbf{0}$ has only the trivial solution.

Defn: If $\mathcal{S}$ is not linearly independent, then it is **linearly dependent**.

Is $\left\{ \begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$ is linearly independent?

$\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ is linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has exactly one solution
(pivot in every column of echelon form of coefficient matrix A).

$\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ is linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.
(at least one free variable)

Thm: Let $\mathcal{S} = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ be a set of vectors in R^k. Then $\mathcal{S}$ is linearly dependent if and only if the vector equation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} = \mathbf{0}$ has an infinite number of solutions.

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$ linearly independent?

$\left\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\}$ is linearly dependent since

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has an infinite number of solutions

$$\left(\text{In particular } \begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{3}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

or equivalently,

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} - \left(\frac{3}{2}\right) \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2) \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

or alternatively,

3 vectors in R^2 cannot be linearly independent.

Is $\{9+7t, 4+8t, 3-5t\}$ linearly independent?

Thm: Let $\mathcal{S}$ be a set of n vectors in R^k where $n > k$. Then $\mathcal{S}$ is linearly dependent.

Thm: A set of vectors is linearly dependent if one of the vectors can be written as a linear combination of the other vectors.

A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{bmatrix}1\\2\\3\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\\2\end{bmatrix}, \begin{bmatrix}5\\7\\9\\1\end{bmatrix}, \begin{bmatrix}6\\2\\1\\5\end{bmatrix}, \begin{bmatrix}4\\3\\4\\6\end{bmatrix}, \begin{bmatrix}1\\4\\3\\7\end{bmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{bmatrix}0\\-1\\-2\end{bmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix} \right\}$ linearly independent?

$$\begin{bmatrix}1&4&5\\2&5&7\\3&6&9\end{bmatrix} \to \begin{bmatrix}1&4&5\\0&-3&-3\\0&-6&-6\end{bmatrix} \to \begin{bmatrix}1&4&5\\0&-3&-3\\0&0&0\end{bmatrix} \blacksquare$$

2.1: Operations on Matrices

$A = (a_{ij}),\ B = (b_{ij}),\ C = (c_{ij})$.

Defn: Two matrices A and B are equal, if they have the same dimension and $a_{ij} = b_{ij}$ for all $i = 1, ..., n,\ j = 1, ..., m$

Defn: $A + B = (a_{ij} + b_{ij})$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} =$

Defn: $cA = (ca_{ij})$.

$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$

Defn: $-B = (-1)B$.

Defn: $A - B = A + (-B)$.

Defn: The zero matrix $= \mathit{0} = (a_{ij})$ where $a_{ij} = 0$ for all i, j.

Ex: $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, ...

Defn: The identity matrix $= I = (a_{ij})$ where $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1$ for all i and I is a square matrix

Ex: $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$.

Suppose A is an $m \times k$ matrix, B is an $k \times n$. $AB = C$ where

$$c_{ij} = row(i)\ of\ A \cdot column(j)\ of\ B$$
$$= a_{i1}b_{1j} + a_{i2}b_{2k} + ... + a_{ir}b_{rk}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} =$$

$$\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} =$$