2.1 cont: Note

$$AB \neq BA$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} =$$

It is also possible that $AB = AC$, but $B \neq C$.

In particular it is possible for $AB = 0$, but $A \neq 0$ AND $B \neq 0$

Defn: If A is a square $(n \times n)$ matrix, $A^0 = I$, $A^1 = A$, $A^k = AA...A$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$

The transpose of the $m \times n$ matrix $A = A^T = (a_{ji})$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T =$

Transpose Properties:

a.) $(A^T)^T = A$

b.) $(A + B)^T = A^T + B^T$

c.) $(kA)^T = kA^T$

d.) $(AB)^T = B^T A^T$

Thm 1 (Properties of matrix arithmetic) Let A, B, C be matrices. Let a, b be scalars. Assuming that the following operations are defined, then

a.) $A + B = B + A$

b.) $A + (B + C) = (A + B) + C$

c.) $A + 0 = A$

d.) $A + (-A) = 0$

e.) $A(BC) = (AB)C$

f.) $AI = A, IB = B$

g.) $A(B + C) = AB + AC,$
. $(B + C)A = BA + CA$

h.) $a(B + C) = aB + aC$

i.) $(a + b)C = aC + bC$

j.) $(ab)C = a(bC)$

k.) $a(AB) = (aA)B = A(aB)$

l.) $1A = A$

Defn.) $-A = -1A$

Cor.) $A0 = 0, 0B = 0$

Cor.) $a0 = 0$

2.2:

Defn: A is invertible if there exists a matrix B such that $AB = BA = I$, and B is called the inverse of A. If the inverse of A does not exist, then A is said to be singular.

Note that if A is invertible, then A is a square matrix.

Thm: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex: The inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is

since

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Thm: Let A be a square matrix. If there exists a square matrix B such that $AB = I$, then $BA = I$ and thus $B = A^{-1}$

Thm: If A is invertible, then its inverse is unique.

Proof: Suppose $AB = I$ and $CA = I$. Then, $B = IB = CAB = CI = C$.

Defn: $A^0 = I$, and if n is a positive integer $A^n = AA \cdots A$ and $A^{-n} = A^{-1}A^{-1} \cdots A^{-1}$.

Thm: If r, s integers, $A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Thm: If A^{-1} and B^{-1}exist, then

i.) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

ii.) A^{-1} is invertible and $(A^{-1})^{-1} = A$

iii.) A^r is invertible and $(A^r)^{-1} = (A^{-1})^r$
where r is any integer

iv.) For any nonzero scalar k,
kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

v.) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$