1.1, 1.2 Solving systems of linear equations.

Example: Solve

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + 5y + 5z = 4 \\ -x - 3y - 2z = -4 \end{cases}$$

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$

$$\underbrace{\begin{bmatrix} \underbrace{\begin{matrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ -1 & -3 & -2 \end{matrix}}_{\text{coef}} & \begin{matrix} 0 \\ 4 \\ -4 \end{matrix} \end{bmatrix}}_{\text{augmente}}$$

↓ eqn 2 - 2 eqn 1 → eqn 2,
↓ eqn 3 + eqn 1 → eqn 3

$$\begin{cases} x + 2y + 3z = 0 \\ 0x + y - z = 4 \\ 0x - y + z = -4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

↓ eqn 3 + eqn 2 → eqn 3

$$\begin{cases} x + 2y + 3z = 0 \\ 0x + y - z = 4 \\ 0x + 0y + 0z = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

opt

depend on problems

↓ eqn 1 - 2eqn 2 → eqn 1

$$\boxed{\begin{matrix} x + 0y + 5z = -8 \\ 0x + y - z = 4 \end{matrix}}$$

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -8 \\ 0 & 1 & -1 & 4 \end{array}\right]$$

$$x + 5z = -8$$
$$y - z = 4$$

Thus $x = -8 - 5z$
$y = 4 + z$
z is free.

z is any real #

System of Linear Equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Coefficient Matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Augmented Matrix Form:

$$\begin{array}{cccccc} x_1 & x_2 & & x_n & & \text{cons} \end{array}$$
$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Elementary row operations:

$$R_i \to cR_i, \quad c \neq 0 \leftarrow \text{NOTE}$$

$$R_i \leftrightarrow R_j$$

$$R_i \to R_i + cR_j$$

Two systems of equations are *equivalent* if they both have the same solution set.

If two augmented matrices are row-equivalent, the corresponding linear systems of equations are equivalent.

Methods of solving a system of linear equations:

1.) Put matrix in Echelon Form

2.) Put matrix in Reduced Echelon form

in echelon form

Echelon form (non-unique):

The leftmost nonzero element in each row is called a *leading entry* or *pivot.*

i.) In each column with a leading entry, all entries below the leading entry are zero.

ii.) Each leading entry of a row is to the left of the leading entry of any row below it.

iii.) All rows of all zeros are below all non-zero rows.

(Note in echelon form, I do not require that the leading entry equal 1)

The position of a leading entry is called the *pivot position.*

A *pivot column* is a column containing a leading entry.

The variable corresponding to a pivot column is called a *basic variable.*

Variables that do not correspond to a pivot column are called *free variables.*

Determine if the augmented matrix is in echelon form. If it is, determine if the corresponding system of equations has no solution, exactly one solution, or an infinite number of solutions. If it has an infinite number of solutions, state the dimension of the hyperplane of the solutions.

L R

EF

$$\begin{bmatrix} 5 & 6 & 3 & 7 & 2 \\ 0 & 7 & 5 & 2 & 8 \\ 0 & 0 & 4 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 0 & 7 & 5 & 2 & 8 \\ 0 & 0 & 4 & 1 & 4 \end{bmatrix}$$

EF

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

B

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

EF

EF

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EF

$$\begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

EF

$$\begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EF

$$\begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

EF

$$\begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EF

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 7 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not EF

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 8 \end{bmatrix}$$

not EF

Row-reduced echelon form (unique):

i.) The matrix is in echelon form.

ii.) The leading entries are all equal to 1.

iii.) In each column with a leading entry, all other entries are zero.

REQUIRED METHOD:

1 To put a matrix in echelon form work from left to right.

2 To put a matrix in row-reduced echelon form:

i.) First put in echelon form (work from left to right).

ii.) Put into reduced echelon form (work from right to left).

You may take short-cuts, but if your method is longer than the above, you will be penalized grade-wise.

Every matrix can be transformed by a finite sequence of elementary row operations into one that is in row-reduced echelon form.

Echelon form is not unique, but row-reduced echelon form is unique.

A system of LINEAR equations can have

i.) No solutions (inconsistent)

ii.) Exactly one solution

iii.) Infinite number of solutions.

free variable $\approx$ non pivot column

Determine if the augmented matrix is in reduced echelon form. If it is, solve.

REF a.)

$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 2 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

$$\begin{bmatrix} 0 & 6 & 0 & 7 & 2 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

REF b.)

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

EF not ref.

c.)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RED

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not ref

$x_1 = -x_2 - x_5 + 2$

x_2 is free

$x_3 = 8$

$x_4 = -5x_5 + 4$

d)

$$\begin{array}{ccccccc} & & x_3 & x_4 & & & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 0 & 5 & 7 & 0 & 0 & 2 \\ 0 & 1 & 3 & 4 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

ref

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ref

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 3 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ref

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ref

$$\begin{bmatrix} 0 & 0 & 1 & 7 & 2 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not ref

$$\begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EF not REF

a.) $x + 5w = 2$
$y + 2w = 8$
$z + w = 4$
$\Rightarrow$ $\boxed{\begin{array}{l} x = 2 - 5w \\ y = 8 - 2w \\ z = 4 - w \\ w \text{ is free} \end{array}}$

under-determined system

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b) ~~x_1 is free~~
no sol'n

c) $\left.\begin{array}{r} x_1 + x_2 \quad + x_5 = 2 \\ x_3 \quad = 8 \\ x_4 + 5x_5 = 4 \\ 0 = 0 \end{array}\right\}$

$x_1 = -x_2 - x_5 + 2$
x_2 is free
$x_3 = 8$
$x_4 = -5x_5 + 4$
x_5 is free

d) $x_1 = -5x_3 - 7x_4 + 2$

$x_2 = -3x_3 - 4x_4 + 8$

x_3 free

x_4 free

$x_5 = 4$

$x_6 = 0$

True/False

Two matrices are row equivalent if they have the same number of rows.

False

$$[0] \ , \ [1]$$

have the same # of rows but are not row equiv

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Two equivalent linear systems can have different solution sets.

False

2 lin sys are equiv iff they have the same soln set

By def'n.