Solve:

$$\left.\begin{aligned} 3x + 6y + 9z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 0 \end{aligned}\right]$$

1.5: A system of equations is **homogeneous** if $b_i = 0$ for all i.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{bmatrix}$$

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

b.) Infinite number of solutions (including, of course, $\mathbf{x} = \mathbf{0}$).

Solve:

$$\begin{aligned} 3x + 6y + 9z &= b_1 \\ 4x + 5y + 6z &= b_2 \\ 7x + 8y + 9z &= b_3 \end{aligned}$$

where 1a.) $b_1 = 0,\ b_2 = 0,\ b_3 = 0$

1b.) $b_1 = 0,\ b_2 = 3,\ b_3 = 0$

1c.) $b_1 = 6,\ b_2 = 5,\ b_3 = 8$

$$\left[\begin{array}{ccc|c} -3 & -6 & -9 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

$\boxed{\begin{array}{l} -R_1 \to R_1 \\ R_1 + R_2 \to R_1 \end{array}}$

$\downarrow \frac{1}{3} R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4^{-4} & 5^{-8} & 6^{-12} & 3^{-0} \\ 7^{-7} & 8^{-14} & 9^{-21} & 0^{-0} \end{array}\right]$$

$\downarrow$ $R_2 - 4R_1 \to R_2$
$R_3 - 7R_1 \to R_3$

$$\left[\begin{array}{cccc} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-3} & -6 & 3 \\ 0 & -6^{+6} & -12^{+12} & 0^{-6} \end{array}\right]$$

$\downarrow$ $R_3 - 2R_2 \to R_3$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right] \leftarrow 0x + 0y + 0z = -6$$

$\boxed{\text{No sol'n}}$

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 & 0 & 2 \\ 0 & 1 & 3 & 4 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e

$$\begin{array}{c} \left[\begin{array}{ccccccc|c} 1 & 4 & 0 & 0 & 3 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccccc} & x_2 & & & x_5 & x_6 & & \end{array} \end{array}$$

free variable

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \end{array}$$

f

$$\begin{bmatrix} 0 & 0 & 1 & 7 & 2 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free variables correspond to non-pivot columns in coeffient matrix

f) $x_1 + \overset{-3x_3}{3x_3} = \overset{-3x_3}{2}$

$x_2 + \overset{-5x_3}{5x_3} = \overset{-5x_3}{8}$

x_3 is free

$$\boxed{\begin{aligned} x_1 &= 2 - 3x_3 \\ x_2 &= 8 - 5x_3 \\ &x_3 \text{ is free} \end{aligned}}$$

e) $x_1 = -4x_2 - 3x_5 - 7x_6 + 2$

x_2 is free

$x_3 = -5x_5 - 2x_6 + 8$

$x_4 = -3x_5 - x_6 + 4$

x_5, x_6 is free

$x_7 = 0$

Row-reduced echelon form (unique):

i.) The matrix is in echelon form.

ii.) The leading entries are all equal to 1.

iii.) In each column with a leading entry, all other entries are zero.

REQUIRED METHOD:

1) To put a matrix in echelon form work from left to right.

2) To put a matrix in row-reduced echelon form:

i.) First put in echelon form (work from left to right). $\longrightarrow$

ii.) Put into reduced echelon form (work from right to left). $\longleftarrow$

You may take short-cuts, but if your method is longer than the above, you will be penalized grade-wise.

Every matrix can be transformed by a finite sequence of elementary row operations into one that is in row-reduced echelon form.

Echelon form is not unique, but row-reduced echelon form is unique.

A system of LINEAR equations can have

i.) No solutions (inconsistent) $\longleftarrow$

ii.) Exactly one solution $\leftarrow$ no free variable

iii.) Infinite number of solutions. $\leftarrow$ free variable

can determine from E.F.

Determine if the augmented matrix is in echelon form. If it is, determine if the corresponding system of equations has no solution, exactly one solution, or an infinite number of solutions. If it has an infinite number of solutions, state the dimension of the hyperplane of the solutions.

1 f.v
∞

$$\left[\begin{array}{cccc|c} 5 & 6 & 3 & 7 & 2 \\ 0 & 7 & 5 & 2 & 8 \\ 0 & 0 & 4 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 7 & 5 & 2 & 8 \\ 0 & 0 & 4 & 1 & 4 \end{array}\right]$$

not EF

$0x_4 = 4$

1 f.v.
$0 = 4$
no sol'n
pivot in constants column

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

1 f.v. $\Rightarrow \infty$

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

∞
1 free variable

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$5x_4 = 0 \Rightarrow x_4 = 0$

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1 unique sol'n

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

$0 = 3$
no soln

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

no f.v.

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 7 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

not EF

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 8 \end{array}\right]$$

not EF

No soln $\Longleftrightarrow$ pivot in last column of augmented matrix

Solve:

$$\begin{aligned} 3x+6y+9z&=0\\ 4x+5y+6z&=3\\ 7x+8y+9z&=0 \end{aligned}$$

1.5: A system of equations is **homogeneous** if $b_i = 0$ for all i.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0\\ a_{21} & a_{22} & \cdots & a_{2n} & 0\\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

b.) Infinite number of solutions (including, of course, $\mathbf{x} = \mathbf{0}$).

Solve:

$$\begin{aligned} 3x+6y+9z&=b_1\\ 4x+5y+6z&=b_2\\ 7x+8y+9z&=b_3 \end{aligned}$$

where 1a.) $b_1 = 0,\ b_2 = 0,\ b_3 = 0$

1b.) $b_1 = 0,\ b_2 = 3,\ b_3 = 0$ ← no sol'n

1c.) $b_1 = 6,\ b_2 = 5,\ b_3 = 8$

solving 3 systems at once

$$\left[\begin{array}{ccc|ccc} \underbrace{3 \quad 6 \quad 9}_{} & & & & & \\ \end{array}\right]$$

$$\overbrace{\begin{matrix}\\\end{matrix}}^{\text{coeff}} \qquad \overbrace{\begin{matrix}\\\end{matrix}}^{\text{constants}}$$

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 9 & 0 & 0 & 6\\ 4 & 5 & 6 & 0 & 3 & 5\\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

$\downarrow \frac{1}{3}R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2\\ 4 & 5 & 6 & 0 & 3 & 5\\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2\\ 0 & -3 & -6 & 0 & 3 & -3\\ 0 & -6 & -12 & 0 & 0 & -6 \end{array}\right]$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2\\ 0 & -3 & -6 & 0 & 3 & -3\\ 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

1a) $X_1 = X_3$
$X_2 = -2X_3$
X_3 is free

$\downarrow$ already know sol'n to system b.

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 2\\ 0 & -3 & -6 & 0 & -3\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1a 1c

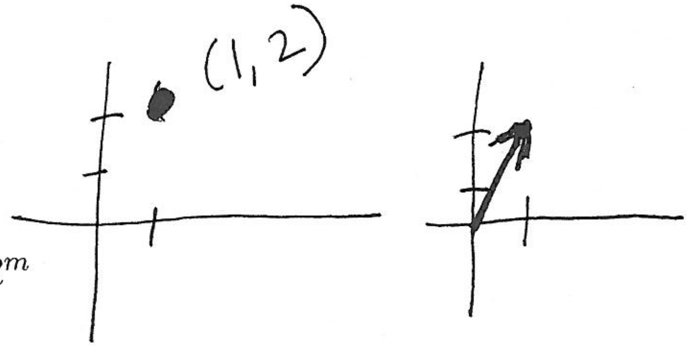
$\downarrow -\frac{1}{3}R_2 \to R_2$

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 0 & 2\\ 0 & 1 & 2 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \xrightarrow{R_1 - 2R_2 \to R_1} \quad \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 0 & 0\\ 0 & 1 & 2 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

 1c

$X_1 = X_3$
$X_2 = -2X_3 + 1$
X_3 is free

1.3 Vectors in R^m



Defn: $\mathbf{u} = (u_1, ..., u_m)$, $\mathbf{v} = (v_1, ..., v_m)$ are **vectors in $\mathbf{R^m}$**.

Defn: $u_1, ..., u_m$ are the **components** of $\mathbf{u}$.

Defn: $\mathbf{u} = \mathbf{v}$ if and only if $u_i = v_i$ for all i.

Defn: The **zero vector** in $\mathbf{R^m}$ is the m-vector $\mathbf{0} = (0, 0, ..., 0)$.

Vector Addition

Defn: The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = (u_1 + v_1, ..., u_m + v_m)$.

Defn: The **negative** of $\mathbf{u}$ is the vector $-\mathbf{u} = (-u_1, ..., -u_m)$

Defn: The **difference** between $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, ..., u_m - v_m)$.

Defn: In this class a **scalar**, c, is a real number.

Defn: The **scalar multiple** of $\mathbf{u}$ by c is the vector $c\mathbf{u} = (cu_1, ..., cu_m)$.

Thm: The vectors, $\mathbf{u}$ and $\mathbf{v}$, are collinear iff there exists a scalar c such that $\mathbf{v} = c\mathbf{u}$. In this case

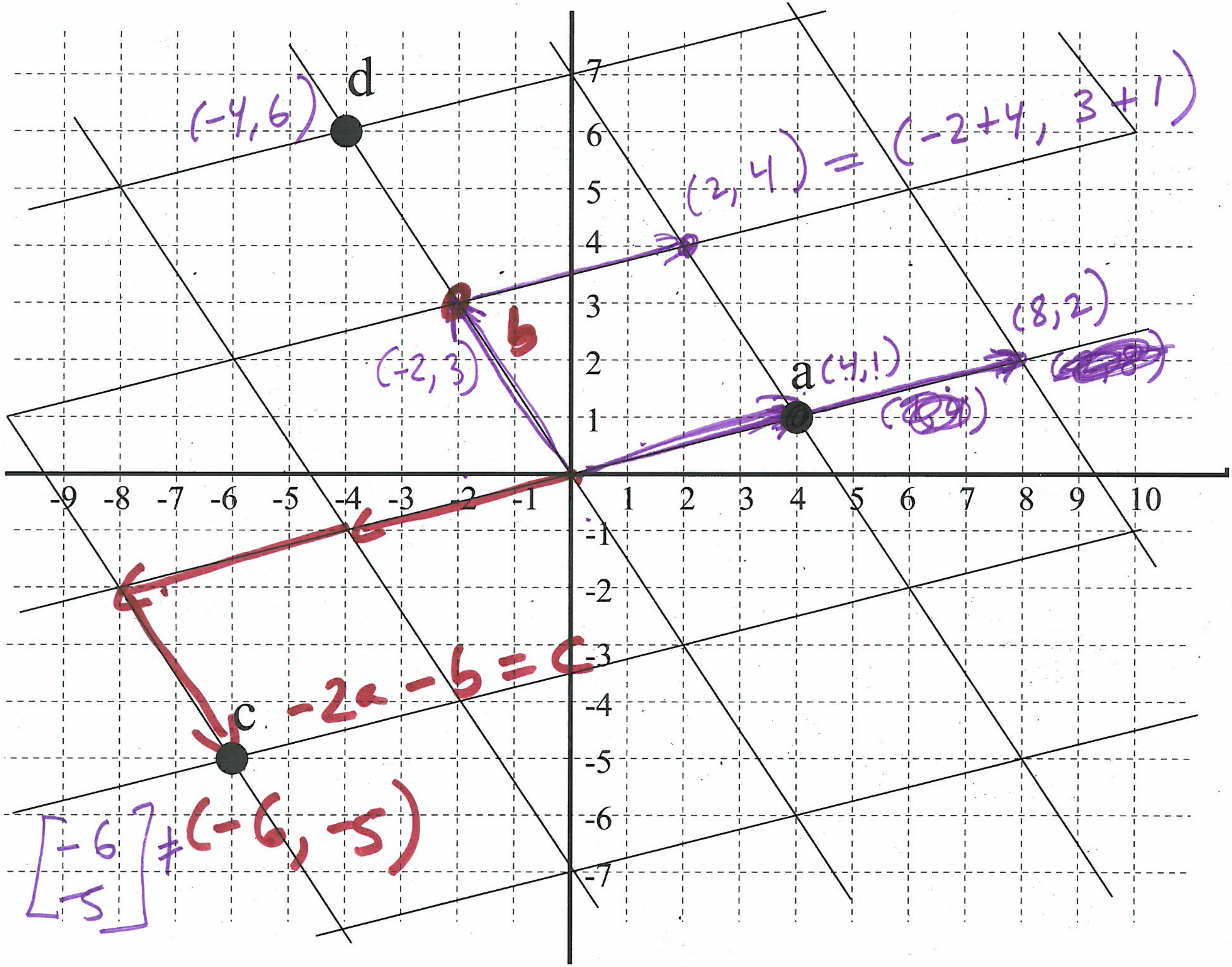
a.) if $c > 0$, $\mathbf{u}$ and $c\mathbf{u}$ have the same direction.

b.) If $c < 0$, $\mathbf{u}$ and $c\mathbf{u}$ have opposite directions.

Defn: The *length (norm, magnitude)* of $\mathbf{u}$ is its distance from $\mathbf{0}$ and is denoted by

$$||\mathbf{u}|| = d(\mathbf{0}, \mathbf{u}) = \sqrt{u_1^2 + u_2^2 + ... + u_m^2}.$$

Two vectors are equivalent if they have the same direction and length.


d
(-4,6)
(2,4) = (-2+4, 3+1)
b
(-2,3)
a (4,1)
(8,2)
c
-2a - b = c
[-6, -5] ≠ (-6, -5)

Parallelogram rule:

Addition: the directed line segment starting at $\mathbf{u}$ and ending at $\mathbf{u}+\mathbf{v}$ is equivalent to $\mathbf{v}$

Subtraction: the directed line segment starting at $\mathbf{u}$ and ending at $\mathbf{v}$ is equivalent to $\mathbf{v}-\mathbf{u}$

Thm 3.2.1 (or thm 4.1.1 p163)

a.) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

b.) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

c.) $\mathbf{u}+\mathbf{0}=\mathbf{u}$

d.) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

e.) $(cd)\mathbf{u}=c(d\mathbf{u})$

f.) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

g.) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

h.) $1\mathbf{u}=\mathbf{u}$

Note $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \neq [x_1 \quad \ldots \quad x_n]$

However, we will sometimes abuse notation.

Sometimes we will write the vector $\mathbf{x}$ as a row vector: $(x_1, \ldots, x_n)$.

Other times we will write the vector $\mathbf{x}$ as a column vector: $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$