$$\underbrace{\begin{bmatrix} -3/2 & -1/2 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}}_{} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3/2 & -1/2 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$$

---

$$\cancel{A^{-1}A}\, x = A^{-1} b$$

$$x = A^{-1} b$$

a unique sol'n

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 2 \\ 6 & 7 & 9 & 1 \end{array}\right]$$

$$\xrightarrow{\quad} \xrightarrow{\text{row ops}} \xrightarrow{\quad}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

A square

A is invertible

$\iff$ A row equiv $I$

$\iff$ $Ax=b$ has a unique sol'n

Elementary matrices and linear systems.

Definition: A matrix is called an elementary matrix if it can be obtained from an identity matrix by exactly one elementary row operation.

Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow kR_1]{} \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[1R_2 + kR_1 \leftrightarrow 1R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cancel{R_2 + kR_1 \to R_1}$ ← not an elem row op

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & j \end{bmatrix}$$

Multiply by elem matrix is equiv to perform elementary row op

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow[R_1 \leftrightarrow kR_1]{} \begin{bmatrix} ka & kb & kc \\ d & e & f \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ d & e & f \\ g & h & j \end{bmatrix}$$

---

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \xrightarrow[R_2 + kR_1 \leftrightarrow R_2]{} \begin{bmatrix} a & b & c \\ d+ka & e+kb & f+kc \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} a & b & c \\ ka+d & kb+e & kc+f \\ g & h & j \end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\xleftrightarrow[R_1\leftrightarrow R_2]{}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Note the opposite of exchanging row one with row two is exchanging row two with row one. That is

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

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$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow[kR_1\Rightarrow R_1]{\cancel{R_1\leftrightarrow kR_1}}\begin{bmatrix}k&0&0\\0&1&0\\0&0&1\end{bmatrix}\xleftrightarrow[\frac{R_1}{k}\to R_1]{}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Note the opposite of multiplying row one by $k$ is multiplying row one by $\frac{1}{k}$. That is,

$$\begin{bmatrix}\frac{1}{k}&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}k&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \begin{bmatrix}k&0&0\\0&1&0\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1/k&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

---

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{R_2+kR_1\leftrightarrow R_2}\begin{bmatrix}1&0&0\\k&1&0\\0&0&1\end{bmatrix}\xleftrightarrow[R_2-kR_1\to R_2]{}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Note the opposite of adding k row 1 to row 2 is subtracting k row 1 from row 2. That is,

$$\begin{bmatrix}1&0&0\\-k&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\k&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \begin{bmatrix}1&0&0\\k&1&0\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\-k&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

Can easily find inverse of elementary matrix

Note $A \sim I$ if and only if $A$ is invertible.

$$\underbrace{(E_n \cdots E_2 E_1)}A = I$$

where $E_i$ are elementary matrices

$$\cancel{E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1} E_n^{-1} E_n E_{n-1} \cdots E_2 E_1 A}$$

$$\cancel{= E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1} E_n^{-1} I}$$

$$A^{-1} A = I$$

$$\boxed{A^{-1} = E_n E_{n-1} \cdots E_2 E_1}$$

Thm 8′: If $A$ is a $\boxed{\text{SQUARE}}$ $n \times n$ matrix, then the following are equivalent.

a.) $A$ is invertible.

b.) The row-reduced echelon form of $A$ is $I_n$, the identity matrix.

c.) An echelon form of $A$ has $n$ leading entries [I.e., every column of an echelon form of $A$ is a leading entry column – no free variables]. (A square => $A$ has leading entry in every column if and only if $A$ has leading entry in every row).

d.) The column vectors of $A$ are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any $b$.

g.) $Ax = b$ has a unique sol'n for any $b$.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix $b$.

i.) $Ax = b$ has at least one sol'n for any $b$.

j.) The column vectors of $A$ span $R^n$. [every vector in $R^n$ can be written as a linear combination of the columns of $A$].

k.) There is a square matrix $C$ such that $CA = I$.

l.) There is a square matrix $D$ such that $AD = I$.

m.) $A^T$ is invertible.

n.) $A$ is expressible as a product of elementary matrices.

# rows = # of columns

square

$a \Rightarrow b$

$A$ invertible

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$\Rightarrow x = A^{-1}b$

$\Rightarrow$ unique sol'n

$[A \mid b] \sim \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & \vdots \\ 0 & & 1 & \end{array}\right]$

$\Rightarrow A \sim I$

$b \Rightarrow c$

$A_{n\times n} \sim I_{n\times n} \Rightarrow n$ pivots ✓

$c \Leftrightarrow d \Leftrightarrow e \Leftrightarrow f$

(leading entry in every column) $\Leftrightarrow$ no free variables

$\Updownarrow$

(c) $\Leftrightarrow$ (e)

$(Ax = 0 \Rightarrow x = 0)$ (e)

$\Leftrightarrow (Ax = b \Rightarrow$ no soln or 1 soln (f)

(don't have $\infty$ # of solns

$A = [\vec{a}_1\ \vec{a}_2 \ldots \vec{a}_n]$

$\Leftrightarrow \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = 0 \Leftrightarrow x_i = 0$ for all $i$

(d) defn of l. i

$d \Leftrightarrow e$

lin ind

$$\vec{a}_1 x_1 + \dots + \vec{a}_n x_n = \vec{0}.$$

$$\begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

$$Ax = 0$$

$$\begin{tikzcd} c \arrow[r, Leftrightarrow] \arrow[rrr, bend right, Leftrightarrow] & h \arrow[r, Leftrightarrow] & i \arrow[r, Leftrightarrow] & j \end{tikzcd}$$

leading entry in every row

$\iff$ no rows of all zeros in EF

$\iff$ $Ax = b$ is consistent given any $b$

since don't have

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & \# \end{array}\right] \quad \# \neq 0$$

If have pivot in every row can solve for pivot variables in terms of free variables

(h) $Ax = b$ is consistent

$\Longleftrightarrow$ $Ax = b$ has at least one sol'n

i)

j) columns of $A$ span $\mathbb{R}^n$

$b$ is in the span of the columns of $A$ $\Longleftrightarrow$ $\vec{a}_1 x_1 + \cdots + \vec{a}_n x_n = \vec{b}$ has a sol'n

$\Longleftrightarrow$ $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$

$A \vec{x} = \vec{b}$ has a sol'n

k)

$$CA = I$$

$$CAx = Cb$$

$$x = Cb$$

$\Rightarrow$ unique sol'n

l)

$$AD = I$$