$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 2 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix}$$

$$|| \ (1R_2 - R_1 \to 1R_2), (1R_4 - 2R_1 \to 1R_4)$$

$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & -4 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & -1 & -4 & -2 \end{bmatrix}$$

$$|| \ \boxed{(R_2 \leftrightarrow R_4)}$$

[redacted]

$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$|| \ (1R_3 + 3R_2 \to 1R_3)$$

does not change determinant

$$-det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & -14 & -5 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$|| \ \boxed{(\tfrac{-1}{14} R_3 \to R_3)}$$

row op: divided by -14

$$-(-14) det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

compensated by multiplying by -14

$$|| \ (R_3 + 4R_4 \to R_4)$$

$$14 det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & 0 & \frac{3}{7} \end{bmatrix} = 14(2)(-1)(1)(\tfrac{3}{7}) = -12$$

$$\begin{vmatrix} 2^{\cdot 0} & 1^{-6} & 3^{+4} & 2^{-2} \\ 2^{-0} & 1^{-3} & -1^{\times 2} & 1^{-1} \\ 0 & 3 & -2 & 1 \\ 4^{-2} & 1^{-1} & 2^{-3} & 2^{-2} \end{vmatrix}$$

$$\left.\begin{array}{l} 1R_4 - R_1 \to 1R_4 \\ 1R_1 - 2R_3 \to 1R_1 \\ 1R_2 - R_3 \to 1R_2 \end{array}\right\} \text{no change in determinat}$$

$$\begin{vmatrix} 2 & -5 & 7 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 3 & -2 & 1 \\ 2 & 0 & -1 & 0 \end{vmatrix}$$

$$= -0 + 0 - 1 \begin{vmatrix} 2 & -5 & 7 \\ 2 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} + 0$$

/3

$$(-1)\left[2\begin{vmatrix}-5 & 7\\ -2 & 1\end{vmatrix} - 0 + (-1)\begin{vmatrix}2 & -5\\ 2 & -2\end{vmatrix}\right]$$

$$= -1\left[2(-5+14) - 1(-4+10)\right]$$

$$= -1\left[18 - 6\right] = -12$$

Method 2

$$-\begin{vmatrix} 2 & -5 & 7\\ 2^{-2} & -2^{+5} & 1^{-7}\\ 2^{-2} & 0^{+5} & -1^{-7}\end{vmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} -\begin{vmatrix} 2 & -5 & 7\\ 0 & 3 & -6\\ 0 & 5 & -8\end{vmatrix}$$

$$= -2\begin{vmatrix}3 & -6\\ 5 & -8\end{vmatrix} = -2(-24+30)$$

$$= -12$$

$\det \begin{bmatrix} 1 & 3 & 9 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$0 \Rightarrow$ free variable

$\det = 0$

$\Leftrightarrow$ free variable

$\Leftrightarrow$ row of all zeros in echelon form

$\Leftrightarrow$ not unique soln

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $det A = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Cor: $det(I_n) = 1$.

Thm: If a square matrix has a row or column containing all zeros, its determinant is zero.

Thm: If some row (column) of a square matrix A is a scalar multiple of another row (column), then $det A = 0$.

Thm: A square matrix is invertible if and only if $det A \neq 0$.

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $det A \neq 0$.

pivot in every row & column

$\Leftrightarrow$ unique sol'n

$\Leftrightarrow$ A is invertible

Thm: $det AB = (det A)(det B)$.

Cor: $det A^{-1} = \frac{1}{det A}$.

$det(A + B) \neq det A + det B$.

Thm: $det A^T = det A$.

Proof of thm $det AB = (det A)(det B)$:

Lemma 1:
Let M be a square matrix, and let E be an elementary matrix of the same order. Then $det(EM) = (det E)(det M)$.

Lemma 2: Let M be a square matrix, and let $E_1, E_2, ..., E_k$ be elementary matrices of the same order as M. Then $det(E_1E_2...E_kM) = (det E_1)(det E_2)...(det E_k)(det M)$.

Lemma 3:
Let $E_1, E_2, ..., E_k$ be elementary matrices of the same order. Then $det(E_1E_2...E_k) = (det E_1)(det E_2)...(det E_k)$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$$\det(AB) = (\det A)(\det B)$$

$$\det(AA^{-1}) = \det I$$

$$= \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = \underline{1}$$

$$\det(AA^{-1}) = \det A \det A^{-1}$$

$$\det A \det A^{-1} = 1$$

$$\boxed{\det A^{-1} = \frac{1}{\det A}}$$

Note A invertible

$$\Longleftrightarrow \det A \neq 0$$

$$\det(A+B) \neq \det A + \det B$$

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

$$|A+B| = \begin{vmatrix} 4 & 4 \\ 0 & 1 \end{vmatrix} = 4$$

$$|A| + |B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 0 \end{vmatrix}$$

$$= 1 + 0 = 1$$

$$4 \neq 1$$

$$|A+B| \neq |A| + |B|$$

$$det\begin{bmatrix} 2 & 1 & 3 & 2 \\ 2 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix}$$

$$\| \ (R_2 - R_1 \to R_2), (R_4 - 2R_1 \to R_4)$$

$$det\begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & -4 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & -1 & -4 & -2 \end{bmatrix} = (-1)^{1+1} 2 det \begin{bmatrix} 0 & -4 & -1 \\ 3 & -2 & 1 \\ -1 & -4 & -2 \end{bmatrix}$$

$$\| \ (R_2 + 3R_3 \to R_3)$$

$$2det\begin{bmatrix} 0 & -4 & -1 \\ 0 & -14 & -5 \\ -1 & -4 & -2 \end{bmatrix}$$

$$\|$$

$$-12 = 2[(-1)\{20 - 14\}] = 2[(-1)^{1+3}(-1)\{(-4)(-5) - (-14)(-1)\}]$$

Suppose $det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 5$ and $det\begin{bmatrix} e & f \\ g & h \end{bmatrix} = 2$

Then $det(4\begin{bmatrix} 3a & c \\ 3b & d \end{bmatrix}\begin{bmatrix} g & h \\ e & f \end{bmatrix}) = det(\begin{bmatrix} 12a & 4c \\ 12b & 4d \end{bmatrix}\begin{bmatrix} g & h \\ e & f \end{bmatrix})$ $R_1/4$ $R_2/4$

$$= det\begin{bmatrix} 12a & 4c \\ 12b & 4d \end{bmatrix} det\begin{bmatrix} g & h \\ e & f \end{bmatrix} = 4^2 det\begin{bmatrix} 3a & c \\ 3b & d \end{bmatrix} det\begin{bmatrix} g & h \\ e & f \end{bmatrix}$$

transpose

$$= 4^2 det\begin{bmatrix} 3a & 3b \\ c & d \end{bmatrix} det\begin{bmatrix} g & h \\ e & f \end{bmatrix} = 3 \times 4^2 det\begin{bmatrix} a & b \\ c & d \end{bmatrix} det\begin{bmatrix} g & h \\ e & f \end{bmatrix}$$ $R_1 \leftrightarrow R_2$

$R_1/3$

$$= -3 \times 4^2 det\begin{bmatrix} a & b \\ c & d \end{bmatrix} det\begin{bmatrix} e & f \\ g & h \end{bmatrix} = -3 \times 4^2 \times 5 \times 2 = -480$$

$$\det\left(4\begin{bmatrix} 3a & c \\ 3b & d \end{bmatrix}\begin{bmatrix} g & h \\ e & f \end{bmatrix}\right)$$

$$= -4^2 \cdot 3 \begin{vmatrix} a & c \\ b & d \end{vmatrix}\begin{vmatrix} e & f \\ g & h \end{vmatrix}$$

$$= -4^2 \cdot 3 \cdot 5 \cdot 2$$

$$= -480$$

2.8 Subspaces of R^n.

Example: The **nullspace of** A is the solution set of $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4]{}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - R_2 \to R_3]{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form

$$\text{Nullspace of } A = \text{Solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

$$= \text{solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

EF

$$= \text{solution space of } \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

REF

$$x_1 + 3x_3 + 4x_4 = 0$$

$$x_2 = 0$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 5 & 6 & 8 & 0 \\ 3 & 7 & 9 & 12 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 - 4x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

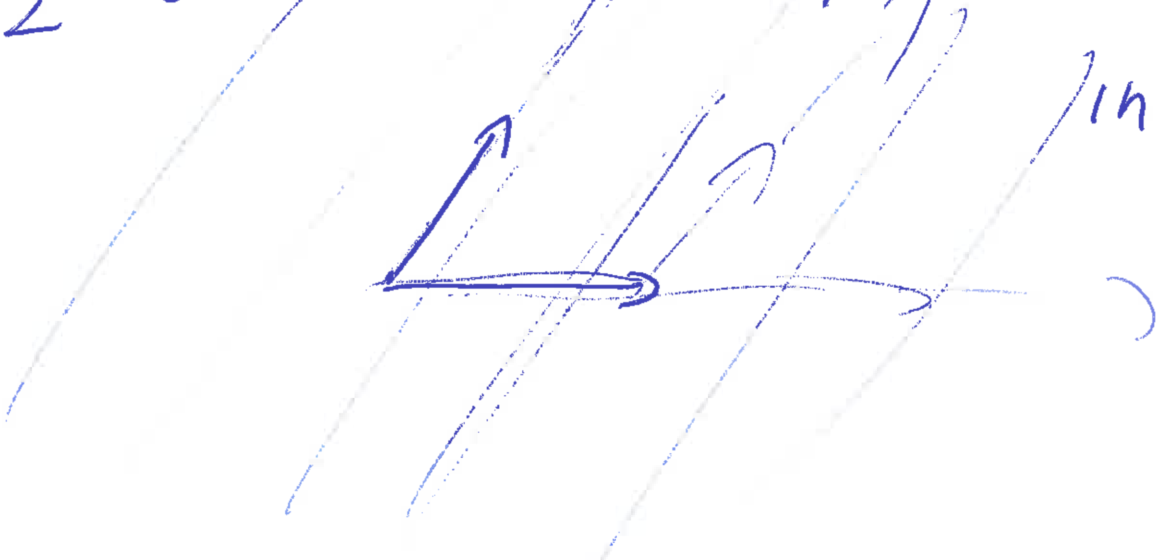
Nullspace of A

$= $ solution set $Ax = 0$

$$= \left\{ x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x_3, x_4 \text{ in } \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$=$ 2-dimensional plane living in $\mathbb{R}^4$



Suppose $A\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} = \mathbf{0}$, then $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$

NOTE: Nullspace of A = span$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.8 Subspaces of R^n.

Long definition emphasizing important points:

Defn: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if and only if the following three conditions are satisfied:

i.) $\mathbf{0}$ is in W,

ii.) if $\mathbf{v_1}, \mathbf{v_2}$ in W, then $\mathbf{v_1} + \mathbf{v_2}$ in W,

iii.) if $\mathbf{v}$ in W, then $c\mathbf{v}$ in W for any scalar c.

Short definition: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if $\mathbf{v_1}, \mathbf{v_2}$ in W implies $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ in W,

---

Note that if S is a subspace, then

if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ in S, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$ is in S.

$0\mathbf{v} = \mathbf{0}$ is in S.

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if

i.) $\mathcal{T}$ is linearly independent and

ii.) $\mathcal{T}$ spans S.