3.3: Cramer's Rule, Adjoint, Inverses, Area

Defn: Let $A_i(\mathbf{b})$ = the matrix derived from A by replacing the i^{th} column of A with $\mathbf{b}$.

Cramer's Rule: Suppose $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix such that $detA \neq 0$. Then

$$x_i = \frac{detA_i(\mathbf{b})}{detA}.$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$$

$$det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = (5)(4) - (6)(2) = 20 - 12 = 8$$

$$det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = (1)(6) - (3)(5) = 6 - 15 = -9$$

Thus $x_1 = \frac{8}{-2} = -4$, $x_2 = \frac{-9}{-2} = \frac{9}{2}$.

Observe for 2×2 case:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} \\ a_{21}x_1 + a_{22}x_2 & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$AI_1(\mathbf{x}) = A_1(\mathbf{b})$$

$$det(AI_1(\mathbf{x})) = det(A_1(\mathbf{b}))$$

$$det(A)\ det(I_1(\mathbf{x})) = det(A_1(\mathbf{b}))$$

$$det(A)\ x_1 = det(A_1(\mathbf{b}))$$

Thus $x_1 = \frac{det(A_1(\mathbf{b}))}{det(A)}$

$$AI_j(\mathbf{x}) = [A\mathbf{e_1}\ \dots\ A\mathbf{e_{j-1}}\ A\mathbf{x}\ A\mathbf{e_{j+1}}\ \dots\ \mathbf{Ae_n}] = A_j(\mathbf{b})$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Defn: For a square matrix A, the (classical) **adjoint** of A is the matrix

$$Adj\,A = [c_{ij}], \text{ where } c_{ij} = (-1)^{i+j} \det A_{ji}.$$

row col'n

In other words, the ij^{th} entry of $Adj\,A$ is the ji^{th} co-factor of A.

Find the adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

remove
j th row
i th col'n

Thm: Let A be a square matrix, with $det A \neq 0$. Then A is invertible, and

$$\boxed{A^{-1} = \frac{1}{det A} Adj\, A.}$$

Proof:
Let $\mathbf{x}$ = the jth column of A^{-1}. Then $A\mathbf{x} = \mathbf{e_j}$

By Cramer's rule, $x_i = \frac{det(A_i(\mathbf{e_j}))}{det(A)}$

$= \text{the } (i,j) \text{ entry of } A^{-1}$

Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

$A_{11} = d \qquad A_{12} = c$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $Adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$det A = ad - bc.$

$A_{21} = b \qquad a_{22} = a$

Thus $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$Adj\ A = \begin{bmatrix} (-1)^{1+1} d & (-1)^{1+2} b \\ (-1)^{2+1} c & (-1)^{2+2} a \end{bmatrix}$

$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \frac{Adj\ A}{det A} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A\vec{X} = \lambda \vec{X}$$

matrix   real #

5.1: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the matrix A if there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is NOT an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A\begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(A - \lambda I)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$.