$P \Rightarrow Q$

If P is true than Q is true

If [$\mathcal{B}$ is a basis for W]

iff ~~then~~ [$W = \text{span}\ \mathcal{B}$ & the vectors in $\mathcal{B}$ are l. indep]

$P \Leftrightarrow Q$

If [$\mathcal{B}$ is a basis for W] then [$W = \text{sp}\ \mathcal{B}$] True

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If [$W = \text{span}\ \mathcal{B}$] the [$\mathcal{B}$ is a basis] False

Also need l. i

[$\mathcal{B}$ is a basis for W because] [$W = \operatorname{span} \mathcal{B}$]

True or False

[If $W = \operatorname{span} \mathcal{B}$] then [$\mathcal{B}$ is a basis]

$$Q \not\Longrightarrow P$$

$$P \Longrightarrow Q$$

Thm 8′: If A is a SQUARE $n \times n$ matrix, then the following are equivalent.

a.) A is invertible.

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries [I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row).

d.) The column vectors of A are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any b.

g.) $Ax = b$ has a unique sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n. [every vector in R^n can be written as a linear combination of the columns of A].

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

m.) A^T is invertible.

n.) A is expressible as a product of elementary matrices.

o.) The column vectors of A form a basis for R^n. [every vector in R^n can be written uniquely as a linear combination of the columns of A].

p.) Col $A = R^n$.

q.) dim Col $A = n$.

r.) rank of $A = n$.

s.) Nul $A = \{\mathbf{0}\}$,

t.) dim Nul $A = 0$.

u.) A has nullity 0.

v.) $\lambda = 0$ is NOT an eigenvalue

$A\vec{x} = \lambda\vec{x}$ has a nonzero sol'n for $\vec{x}$

$A x - \lambda I \vec{x} = 0$

$(A - \lambda I)\vec{x} = \vec{0}$

Rank(A) + nullity(A) = Number of columns of A.

Ex. 2) Suppose A is a $9X4$ matrix.

If Rank(A) = 4, then nullity(A) =

$\mathbf{Ax} = \mathbf{0}$ has ______________ solutions.

$\mathbf{Ax} = \mathbf{b}$ has ______________ solutions.

If Rank(A) = 3, then nullity(A) =

$\mathbf{Ax} = \mathbf{0}$ has ______________ solutions.

$\mathbf{Ax} = \mathbf{b}$ has ______________ solutions.

If $\lambda = 0$ is an e. value of A

$(A - \lambda I)\vec{x} = A\vec{x} = 0$ has an infinite # of sol'ns

5.1: Eigenvalues and Eigenvectors

λ scalar in Eng Math III
$\lambda \in \mathbb{R}$

Defn: λ is an **eigenvalue** of the matrix A if there exists a <u>nonzero</u> vector $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \lambda\mathbf{x}}$.

The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is <u>NOT</u> an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $A\mathbf{x} - \lambda I\mathbf{x} = \boxed{(A - \lambda I)\mathbf{x} = \mathbf{0}}$ $\quad D\mathbf{x} = 0$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $\underbrace{(A - \lambda I)\mathbf{x} = \mathbf{0}}_{\infty \text{ \# of soln}}$ has a nonzero solution.

Thus $det(A - \lambda I)\cancel{\mathbf{x}} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$\boxed{det(A - \lambda I) = 0} \text{ for } \lambda.$$

characteristic equation

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$\boxed{(A - \lambda_0 I)\mathbf{x} = 0 \text{ for } \mathbf{x}.}$$

Defn: $det(A - \lambda I) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(A - \lambda_0 I)\mathbf{x} = \mathbf{0}$.

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Find the e. values & their corresponding e. vectors

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$|A - \lambda I|$

$$= \begin{vmatrix} (1-\lambda) & 2 \\ 3 & (4-\lambda) \end{vmatrix} = (1-\lambda)(4-\lambda) - 6$$

$$= 4 - 5\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 5\lambda - 2 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(-2)}}{2}$$

$$= \frac{5 \pm \sqrt{33}}{2}$$

E. value $\lambda = \dfrac{5 \pm \sqrt{33}}{2}$

Find e. vectors: Solve $(A - \lambda I)\vec{x} = 0$

for nonzero e. vectors $\vec{x}$

$$\left[\begin{array}{cc|c} 1-\left(\frac{5\pm\sqrt{33}}{2}\right) & 2 & 0 \\ 3 & 4-\left(\frac{5\pm\sqrt{33}}{2}\right) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \frac{-3 \mp \sqrt{33}}{2} & 2 & 0 \\ 3 & 4-\left(\frac{5\pm\sqrt{33}}{2}\right) & 0 \end{array}\right]$$

$R_1/(\ )$

$$\left[\begin{array}{cc|c} 1 & 2\left(\frac{2}{-3 \mp \sqrt{33}}\right) & \\ 3 & \frac{+3 \mp \sqrt{33}}{2} & \end{array}\right]$$

simplify first

$$2\left(\frac{2}{-3\mp\sqrt{33}}\right)\left(\frac{-3\pm\sqrt{33}}{-3\pm\sqrt{33}}\right)$$

$$=\frac{4(-3\pm\sqrt{33})}{9-33}$$

$$=\frac{\cancel{4}(-3\pm\sqrt{33})}{-\cancel{24}\ 6}$$

$$=\frac{3\mp\sqrt{33}}{6}$$

$$\begin{bmatrix} 1 & \frac{3 \mp \sqrt{33}}{6} \\ 3 - 3 & \frac{3 \mp \sqrt{33}}{2} - 3\left(\frac{3 \mp 33}{6}\right) \end{bmatrix}$$

$$\xrightarrow{R2 - 3R1} \left[\begin{array}{cc|c} 1 & \frac{3 \mp \sqrt{33}}{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

∞ # of sol'ns

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-3 \pm \sqrt{33}}{6} x_2 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-3 \pm \sqrt{33}}{6} \\ 1 \end{bmatrix} x_2$$

$\lambda_1 = \frac{5+\sqrt{33}}{2}$ has an e. vector of $\begin{bmatrix} \frac{-3+\sqrt{33}}{6} \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{5-\sqrt{33}}{2}$ has an e. vector of $\begin{bmatrix} \frac{-3-\sqrt{33}}{6} \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -3-\sqrt{33} \\ 6 \end{bmatrix}$$

Find e. valu & e. vectors fo

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I)$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 & 4 \\ 0 & 2-\lambda & 6 & 5 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)^3 = 0$$

$$\lambda = 1, 2$$

repeated e. value

Find e. vectors con to e. value $\lambda = 1$

Solve $(A-\lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 0 \\ 0 & 1 & 6 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 - 4R_4$

$R_2 - 5R_4$

$R_1 - 3R_3$

$R_1 - 6R_3$

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$