Thm: Suppose λ_i, $i = 1, ..., n$ are DISTINCT eigenvalues of a matrix A. If $\mathcal{B}_i$ is a basis for the eigenspace corresponding to λ_i, then

$\mathcal{B} = \mathcal{B}_1 \cup ... \cup \mathcal{B}_n$ is linearly independent.

Defn: Suppose the characteristic polynomial of A is

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2}...(\lambda - \lambda_n)^{k_n}$$

where the λ_i, $i = 1, ..., n$ are DISTINCT. Then the **algebraic multiplicity of** λ_i is k_i.

That is the **algebraic multiplicity of** λ_i is the number of times that $(\lambda - \lambda_i)$ appears as a factor of the characteristic polynomial of A.

Defn: The **geometric multiplicity of** λ_i = dimension of the eigenspace corresponding to λ_i.

Thm (Geometric and Algebraic Multiplicity):

a.) The geometric multiplicity is less than or equal to the algebraic multiplicity [That is, Nullity of $(\lambda_i I - A) \leq k_i$].

b.) A is diagonalizable if and only if the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue.

Inner Product Example: Dot product on R^n.

Defn: $\Sigma_{k=1}^m a_k = a_1 + a_2 + ... + a_m$

Defn:
The **dot product** of $\mathbf{u} = (u_1, ..., u_m)$ & $\mathbf{v} = (v_1, ..., v_m)$ is

$$\mathbf{u} \cdot \mathbf{v} = \Sigma_{k=1}^m u_k v_k.$$

In words, $\mathbf{u} \cdot \mathbf{v}$ is the sum of the products of the corresponding components of $\mathbf{u}$ and $\mathbf{v}$.

Note that $\mathbf{u} \cdot \mathbf{v}$ is a real number (not a vector).

Examples:

$(1, 2, 3) \cdot (4, 5, 6) = 4 + 10 + 18 = 32$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 + 2 = 0$

Defn: Let $\mathbf{v}$ be a vector in an inner product space $\mathbf{V}$. The **length** or **norm** of $\mathbf{v} = ||\mathbf{v}|| = \sqrt{<\mathbf{v}, \mathbf{v}>}$. $\sqrt{v \cdot v}$

pyth

$||(3, 4)|| = \sqrt{3^2 + 4^2} = \sqrt{5^2} = 5$

Defn: The vector $\mathbf{u}$ is a **unit vector** if $||\mathbf{u}|| = 1$.

6.1: Inner Products.

Defn: Let V be a vector space over the real numbers. An **inner product** for V is a function that associates a real number $\mathbf{u} \cdot \mathbf{v}$ to every pair of vectors, $\mathbf{u}$ and $\mathbf{v}$ in V such that the following properties are satisfied for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and scalars c:

a.) $\mathbf{u} \cdot \mathbf{v} \;=\; \mathbf{v} \cdot \mathbf{u}$ ✓

b.) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \;=\; \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

c.) $(c\mathbf{u}) \cdot v \;=\; c(\mathbf{u} \cdot \mathbf{v}) \;=\; \mathbf{u} \cdot (c\mathbf{v})$

d.) $\mathbf{u} \cdot \mathbf{u} \;\geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

$u \cdot u = \|u\|^2$

A vector space V together with an inner product is called an **inner product space.**

Thm 6.1.1′: Let V be an inner product space. Then for all vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{v}$ in V and scalars c_1, c_2:

a.) $(c_1\mathbf{u_1} + c_2\mathbf{u_2}) \cdot \mathbf{v} \;=\; \mathbf{v} \cdot (c_1\mathbf{u_1} + c_2\mathbf{u_2})$ ✓

$$= c_1(\mathbf{u_1} \cdot \mathbf{v}) + c_2(\mathbf{u_2} \cdot \mathbf{v})$$

b.) $\mathbf{0} \cdot \mathbf{v} \;=\; \mathbf{v} \cdot \mathbf{0} \;=\; 0$

Note that $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

Normalize $\vec{v} \Rightarrow \frac{\vec{v}}{||v||}$ unit vector

Create a unit vector in the direction of the vector (3, 4):

$||(3,4)|| = 5$

$$\boxed{\left(\frac{3}{5}, \frac{4}{5}\right)}$$

Create a unit vector in the direction of the vector (1, 2):

$||(1,2)|| = \sqrt{1^2+2^2} = \sqrt{5}$

$$\boxed{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)}$$

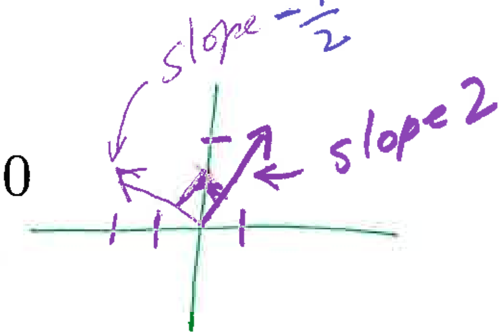
Create a unit vector in the direction of the vector (-2, 1):

$||(-2,1)|| = \sqrt{4+1} = \sqrt{5}$

$$\boxed{\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)}$$

Defn: $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if

$$\boxed{\mathbf{u}\cdot\mathbf{v} = 0.}$$

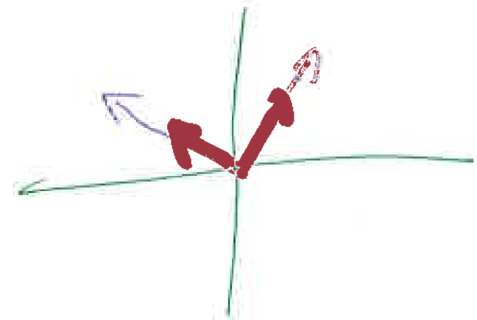


Example: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1(-2) + 2(1) = 0$

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

normalizing

Example: $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = 0$



Thus $\left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

Observation:

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Suppose $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is a pair of orthogonal unit vectors. Then

$$\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

orthonormal $\qquad U^T U = I$