1. Let $A = \begin{bmatrix} 1 & 3 & 10 & 6 \\ -3 & 7 & 0 & 6 \\ -2 & 2 & -5 & 0 \end{bmatrix}$

[8] **1a.)** Find a basis for the column space of A: $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 10 & 6 \\ -3 & 7 & 0 & 6 \\ -2 & 2 & -5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 10 & 6 \\ 0 & 16 & 30 & 24 \\ 0 & 8 & 15 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 10 & 6 \\ 0 & 8 & 15 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 10 & 6 \\ 0 & 1 & \frac{15}{8} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 3 & \frac{80-45}{8} & \frac{12-9}{2} \\ 0 & 1 & \frac{15}{8} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{35}{8} & \frac{3}{2} \\ 0 & 1 & \frac{15}{8} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: To determine a basis for the column space of a matrix, you do NOT need REF. Echelon form will suffice for determining pivot columns, so as soon as you know which columns are pivot columns, you can stop performing row ops IF you are ONLY interested in finding a basis for col(A).

However, we are also interested in solving a system of equations in order to answer question 1e. But you only needed to write one column as a linear combination of the other columns, so you could have taken just 3 of the columns (both pivot columns and 1 free variable column in order to write the free variable column as a linear combination of the pivot columns).

In other words, instead of the last few row op steps above, you could delete, for example, free variable column 3, in order to solve for free variable column 4 in terms of pivot columns 1 and 2.

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{12-9}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

[2] **1b.)** Rank(A) = 2

[2] **1c.)** Nullity(A) =2

[3] **1d.)** Are columns of A linearly independent? no

[5] **1e.)** If possible write one of the columns of A as a linear combination of the other columns of A.

$$\frac{3}{2}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix}$$

or

$$\frac{35}{8}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \frac{15}{8}\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -5 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 12 & 4 & 0 & 0 \\ -30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

[12] **3a).** Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Note, you may use the following facts:

(1.) A has eigenvalue $\lambda_1 = 0$ with multiplicity 1.

(2.) A has eigenvalue λ_2 with multiplicity 3.

(3.) The vector $\begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of A.

Using fact 3, we see that $\lambda_2 = 2$ per the calculation below:

$$\begin{bmatrix} 12 & 4 & 0 & 0 \\ -30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Thus per facts 1 and 2, the characteristic polynomial of $A = \lambda(\lambda - 2)^3$ and $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

For the eigenspace corresponding to $= 2$: $A - 2I =$

$$\begin{bmatrix} 10 & 4 & 0 & 0 \\ -30 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{Thus} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.4x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For the eigenspace corresponding to $= 0$:

$$A - 0I = \begin{bmatrix} 12 & 4 & 0 & 0 \\ -30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3a. $P = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

[3] **3b.** The characteristic polynomial of the matrix $A = \lambda(\lambda - 2)^3$

Alternatively one can solve $det(A-\lambda I)=0$ using cofactor expansion:

$$\begin{bmatrix} 12-\lambda & 4 & 0 & 0 \\ \text{-30} & -10-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda)[(12-\lambda)(-10-\lambda)+120]$$

$$= (2-\lambda)(2-\lambda)[-120-2\lambda+\lambda^2+120] = (2-\lambda)(2-\lambda)[-2\lambda+\lambda^2] = (2-\lambda)(2-\lambda)(-2+\lambda)\lambda$$

Thus we get the same characteristic polynomial and same matrix D as above.

[2] **5.** Circle the correct answer:

Suppose $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ has

- B. Unique solution

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6. Fill in the SIX blanks below:

Suppose that A is a 10×12 matrix which has 4 pivot columns, then

[2] **6a.** The rank of A = 4

[2] **6b.** The nullity of A = 8

[4] **6c.** The column space of A is a 4 dimensional subspace of R^k where $k =$10

[4] **6d.** The nullspace of A is a 8 dimensional subspace of R^n where $n =$12

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7. Circle T for true and F for False

[3] **7a.** If $A = QR$, where Q has **orthonormal** columns, then $R = Q^T A$. T

[3] **7b.** If $A = QR$, where Q has **orthogonal** columns, then $R = Q^T A$. F

[3] **7c.** Suppose the Gram-Schmidt process is applied to the linearly independent set $\{x_1, ..., x_p\}$ to form an orthogonal set $\{v_1, ..., v_p\}$. Then span $\{v_1, ..., v_p\}$ = span $\{x_1, ..., x_p\}$. T

[3] **7d.** Suppose $A = PDP^{-1}$ where D is a diagonal matrix. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$, then $3\vec{p}_2$ is an eigenvector of A T

[2] **7e.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) \leq 4$. T

[2] **7f.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) = 4$. F

[2] **7g.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$ and $\{u_1, u_2, u_3, u_4\}$ is a linearly independent set, then $dim(S) = 4$. T

1. Let $A = \begin{bmatrix} 1 & 4 & 3 & 5 \\ -2 & 2 & -5 & 0 \\ -5 & 0 & -13 & -5 \end{bmatrix}$

[8] **1a.)** Find a basis for the column space of A:______________________________________

$$\begin{bmatrix} 1 & 4 & 3 & 5 \\ -2 & 2 & -5 & 0 \\ -5 & 0 & -13 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 3 & 5 \\ 0 & 10 & 1 & 10 \\ 0 & 20 & 2 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 3 & 5 \\ 0 & 10 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 3 & 5 \\ 0 & 1 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2.6 & 1 \\ 0 & 1 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
)

Note: To determine a basis for the column space of a matrix, you do NOT need REF. Echelon form will suffice for determining pivot columns, so as soon as you know which columns are pivot columns, you can stop performing row ops IF you are ONLY interested in finding a basis for col(A).

However, we are also interested in solving a system of equations in order to answer question 1e. But you only needed to write one column as a linear combination of the other columns, so you could have taken just 3 of the columns (both pivot columns and 1 free variable column in order to write the free variable column as a linear combination of the pivot columns).

In other words, instead of the last few row op steps above, you could delete, for example, free variable column 3, in order to solve for free variable column 4 in terms of pivot columns 1 and 2.

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 10 & 10 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

[2] **1b.)** Rank(A) = 2

[2] **1c.)** Nullity(A) = 2

[3] **1d.)** Are columns of A linearly independent? no

[5] **1e.)** If possible write one of the columns of A as a linear combination of the other columns of A.

$$\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

or

$$2.6 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 0.1 \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ -13 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 12 & \text{-30} \\ 0 & 0 & 4 & \text{-10} \end{bmatrix}$.

[12] **3a).** Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Note, you may use the following facts:

(1.) A has eigenvalue $\lambda_1 = 0$ with multiplicity 1.

(2.) A has eigenvalue λ_2 with multiplicity 3.

(3.) The vector $\begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A.

Using fact 3, we see that $\lambda_2 = 2$ per the calculation below:

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 12 & \text{-30} \\ 0 & 0 & 4 & \text{-10} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Thus per facts 1 and 2, the characteristic polynomial of $A = \lambda(\lambda - 2)^3$ and $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

For the eigenspace corresponding to $= 2$: $A - 2I =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & \text{-30} \\ 0 & 0 & 4 & \text{-12} \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.4x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For the eigenspace corresponding to $= 0$:

$$A - 0I = \begin{bmatrix} 12 & 4 & 0 & 0 \\ \text{-30} & \text{-10} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3a. $P = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

[3] **3b.** The characteristic polynomial of the matrix $A = \lambda(\lambda - 2)^3$

Alternatively one can solve $det(A - \lambda I) = 0$ using cofactor expansion:

$$\begin{bmatrix} 12-\lambda & 4 & 0 & 0 \\ \text{-30} & -10-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda)[(12-\lambda)(-10-\lambda)+120]$$

$$= (2-\lambda)(2-\lambda)[-120-2\lambda+\lambda^2+120] = (2-\lambda)(2-\lambda)[-2\lambda+\lambda^2] = (2-\lambda)(2-\lambda)(-2+\lambda)\lambda$$

Thus we get the same characteristic polynomial and same matrix D as above.

[2] **5.** Circle the correct answer:

Suppose $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ has

- B. Unique solution

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6. Fill in the SIX blanks below:

Suppose that A is a 7×9 matrix which has 6 pivot columns, then

[2] **6a.** The rank of A = 6

[2] **6b.** The nullity of A = 3

[4] **6c.** The column space of A is a 6 dimensional subspace of R^k where k =7

[4] **6d.** The nullspace of A is a 3 dimensional subspace of R^n where n =9

---

7. Circle T for true and F for False

[2] **7a.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) \leq 4$. T

[2] **7b.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) = 4$. F

[2] **7c.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$ and $\{u_1, u_2, u_3, u_4\}$ is a linearly independent set, then $dim(S) = 4$.
T

[3] **7d.** Suppose $A = PDP^{-1}$ where D is a diagonal matrix. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$, then $3\vec{p}_2$ is an eigenvector of A
T

[3] **7e.** If $A = QR$, where Q has **orthonormal** columns, then $R = Q^T A$. T

[3] **7f.** If $A = QR$, where Q has **orthogonal** columns, then $R = Q^T A$. F

[3] **7g.** Suppose the Gram-Schmidt process is applied to the linearly independent set $\{x_1, ..., x_p\}$ to form an orthogonal set $\{v_1, ..., v_p\}$. Then span $\{v_1, ..., v_p\}$ = span $\{x_1, ..., x_p\}$.
T

1. Let $A = \begin{bmatrix} 1 & -4 & 5 & 3 \\ -2 & 2 & -5 & 0 \\ -3 & 0 & -5 & 3 \end{bmatrix}$

[8] **1a.)** Find a basis for the column space of A: $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -4 & 5 & 3 \\ -2 & 2 & -5 & 0 \\ -3 & 0 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 5 & 3 \\ 0 & -6 & 5 & 6 \\ 0 & -12 & 10 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 5 & 3 \\ 0 & 1 & -\frac{5}{6} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{30-20}{6} & -1 \\ 0 & 1 & -\frac{5}{6} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & \frac{10}{6} & -1 \\ 0 & 1 & -\frac{5}{6} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{5}{3} & -1 \\ 0 & 1 & -\frac{5}{6} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: To determine a basis for the column space of a matrix, you do NOT need REF. Echelon form will suffice for determining pivot columns, so as soon as you know which columns are pivot columns, you can stop performing row ops IF you are ONLY interested in finding a basis for col(A).

However, we are also interested in solving a system of equations in order to answer question 1e. But you only needed to write one column as a linear combination of the other columns, so you could have taken just 3 of the columns (both pivot columns and 1 free variable column in order to write the free variable column as a linear combination of the pivot columns).

In other words, instead of the last few row op steps above, you could delete, for example, free variable column 3, in order to solve for free variable column 4 in terms of pivot columns 1 and 2.

$$\begin{bmatrix} 1 & -4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

[2] **1b.)** Rank(A) = 2

[2] **1c.)** Nullity(A) = 2

[3] **1d.)** Are columns of A linearly independent? no

[5] **1e.)** If possible write one of the columns of A as a linear combination of the other columns of A.

$$-\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

or

$$\frac{5}{3}\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} - \frac{5}{6}\begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ -5 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 12 & 30 \\ 0 & 0 & -4 & -10 \end{bmatrix}$.

[12] **3a).** Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Note, you may use the following facts:

(1.) A has eigenvalue $\lambda_1 = 0$ with multiplicity 1.

(2.) A has eigenvalue λ_2 with multiplicity 3.

(3.) The vector $\begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A. Using fact 3, we see that $\lambda_2 = 2$ per the calculation below:

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 12 & -30 \\ 0 & 0 & 4 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Thus per facts 1 and 2, the characteristic polynomial of $A = \lambda(\lambda - 2)^3$ and $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

For the eigenspace corresponding to $= 2$: $A - 2I =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & -30 \\ 0 & 0 & 4 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.4x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For the eigenspace corresponding to $= 0$:

$$A - 0I = \begin{bmatrix} 12 & 4 & 0 & 0 \\ -30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3a. $P = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

[3] **3b.** The characteristic polynomial of the matrix $A = \lambda(\lambda - 2)^3$

Alternatively one can solve $det(A-\lambda I)=0$ using cofactor expansion:

$$\begin{bmatrix} 12-\lambda & 4 & 0 & 0 \\ \text{-30} & -10-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda)[(12-\lambda)(-10-\lambda)+120]$$

$$= (2-\lambda)(2-\lambda)[-120-2\lambda+\lambda^2+120] = (2-\lambda)(2-\lambda)[-2\lambda+\lambda^2] = (2-\lambda)(2-\lambda)(-2+\lambda)\lambda$$

Thus we get the same characteristic polynomial and same matrix D as above.

[2] **5.** Circle the correct answer:

Suppose $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ has

- B. Unique solution

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6. Fill in the SIX blanks below:

Suppose that A is a 5×11 matrix which has 3 pivot columns, then

[2] **6a.** The rank of A = 3

[2] **6b.** The nullity of A = 8

[4] **6c.** The column space of A is a 3 dimensional subspace of R^k where k =5

[4] **6d.** The nullspace of A is a 8 dimensional subspace of R^n where n =11

---

7. Circle T for true and F for False

[3] **7a.** If $A = QR$, where Q has **orthogonal** columns, then $R = Q^T A$. F

[3] **7b.** If $A = QR$, where Q has **orthonormal** columns, then $R = Q^T A$. T

[3] **7c.** Suppose $A = PDP^{-1}$ where D is a diagonal matrix. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$, then $3\vec{p}_2$ is an eigenvector of A

T

[3] **7d.** Suppose the Gram-Schmidt process is applied to the linearly independent set $\{x_1, ..., x_p\}$ to form an orthogonal set $\{v_1, ..., v_p\}$. Then span $\{v_1, ..., v_p\}$ = span $\{x_1, ..., x_p\}$.

T

[2] **7e.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) = 4$. F

[2] **7f.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) \leq 4$. T

[2] **7g.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$ and $\{u_1, u_2, u_3, u_4\}$ is a linearly independent set, then $dim(S) = 4$.
T

1. Let $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ -2 & 10 & 2 & 0 \\ -5 & 17 & 0 & -5 \end{bmatrix}$

[8] **1a.)** Find a basis for the column space of A:____

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ -2 & 10 & 2 & 0 \\ -5 & 17 & 0 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 16 & 10 & 10 \\ 0 & 32 & 20 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 16 & 10 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & \frac{5}{8} & \frac{5}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & \frac{32-15}{8} & \frac{40-15}{8} \\ 0 & 1 & \frac{5}{8} & \frac{5}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{17}{8} & \frac{25}{8} \\ 0 & 1 & \frac{5}{8} & \frac{5}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: To determine a basis for the column space of a matrix, you do NOT need REF. Echelon form will suffice for determining pivot columns, so as soon as you know which columns are pivot columns, you can stop performing row ops IF you are ONLY interested in finding a basis for col(A).

However, we are also interested in solving a system of equations in order to answer question 1e. But you only needed to write one column as a linear combination of the other columns, so you could have taken just 3 of the columns (both pivot columns and 1 free variable column in order to write the free variable column as a linear combination of the pivot columns).

In other words, instead of the last few row op steps above, you could delete, for example, free variable column 4, in order to solve for free variable column 3 in terms of pivot columns 1 and 2.

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 16 & 10 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & \frac{5}{8} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{32-15}{8} \\ 0 & 1 & \frac{5}{8} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{17}{8} \\ 0 & 1 & \frac{5}{8} \\ 0 & 0 & 0 \end{bmatrix}$$

[2] **1b.)** Rank(A) = 2

[2] **1c.)** Nullity(A) = 2

[3] **1d.)** Are columns of A linearly independent? no

[5] **1e.)** If possible write one of the columns of A as a linear combination of the other columns of A.

$$\frac{17}{8}\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \frac{5}{8}\begin{bmatrix} 3 \\ 10 \\ 17 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

or

$$\frac{25}{8}\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \frac{5}{8}\begin{bmatrix} 3 \\ 10 \\ 17 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 12 & -4 & 0 & 0 \\ 30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

[12] **3a).** Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Note, you may use the following facts:

(1.) A has eigenvalue $\lambda_1 = 0$ with multiplicity 1.

(2.) A has eigenvalue λ_2 with multiplicity 3.

(3.) The vector $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ is an eigenvector of A.

Using fact 3, we see that $\lambda_2 = 2$ per the calculation below:

$$\begin{bmatrix} 12 & -4 & 0 & 0 \\ 30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Thus per facts 1 and 2, the characteristic polynomial of $A = \lambda(\lambda-2)^3$ and $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

For the eigenspace corresponding to $= 2$: $A - 2I =$

$$\begin{bmatrix} 10 & -4 & 0 & 0 \\ 30 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.4x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For the eigenspace corresponding to $= 0$:

$$A - 0I = \begin{bmatrix} 12 & 4 & 0 & 0 \\ -30 & -10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Thus } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3a. $P = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

[3] **3b.** The characteristic polynomial of the matrix $A = \lambda(\lambda-2)^3$

Alternatively one can solve $det(A-\lambda I)=0$ using cofactor expansion:

$$\begin{bmatrix} 12-\lambda & 4 & 0 & 0 \\ \text{-30} & -10-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda)[(12-\lambda)(-10-\lambda)+120]$$

$$= (2-\lambda)(2-\lambda)[-120-2\lambda+\lambda^2+120] = (2-\lambda)(2-\lambda)[-2\lambda+\lambda^2] = (2-\lambda)(2-\lambda)(-2+\lambda)\lambda$$

Thus we get the same characteristic polynomial and same matrix D as above.

[2] **5.** Circle the correct answer:

Suppose $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ has

- B. Unique solution

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6. Fill in the SIX blanks below:

Suppose that A is a 8×6 matrix which has 5 pivot columns, then

[2] **6a.** The rank of A = 5

[2] **6b.** The nullity of A = 1

[4] **6c.** The column space of A is a 5 dimensional subspace of R^k where k =8

[4] **6d.** The nullspace of A is a 1 dimensional subspace of R^n where n =6

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7. Circle T for true and F for False

[3] **7a.** Suppose $A = PDP^{-1}$ where D is a diagonal matrix. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$, then $3\vec{p}_2$ is an eigenvector of A

T

[2] **7b.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) = 4$. F

[2] **7c.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$, then $dim(S) \leq 4$. T

[2] **7d.** If $S = \text{span}\{u_1, u_2, u_3, u_4\}$ and $\{u_1, u_2, u_3, u_4\}$ is a linearly independent set, then $dim(S) = 4$.

T

[3] **7e.** If $A = QR$, where Q has **orthogonal** columns, then $R = Q^T A$. F

[3] **7f.** If $A = QR$, where Q has **orthonormal** columns, then $R = Q^T A$. T

[3] **7g.** Suppose the Gram-Schmidt process is applied to the linearly independent set $\{x_1, ..., x_p\}$ to form an orthogonal set $\{v_1, ..., v_p\}$. Then span $\{v_1, ..., v_p\}$ = span $\{x_1, ..., x_p\}$.

T