MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 20, 2013: Persistent homology.

Fall 2013 course offered through the University of Iowa Division of Continuing Education

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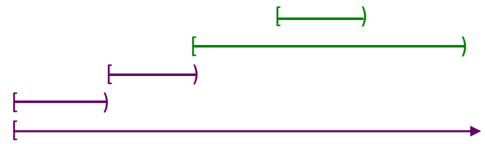
Applied Mathematical and Computational Sciences, University of Iowa

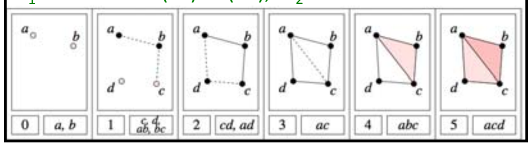
http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

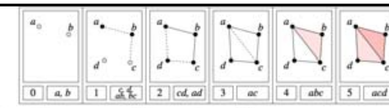
$H_0 = \langle a, b, c, d \;:\; tc + td,\ tb + c,\ ta + tb \rangle$

$H_1 = \langle z_1, z_2 \;:\; t z_2,\ t^3 z_1 + t^2 z_2 \rangle$



$z_1 = ad + cd + t(bc) + t(ab)$, $z_2 = ac + t^2 bc + t^2 ab$



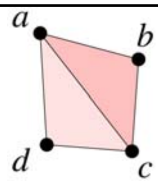
$$\begin{array}{ccccccc} C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \cdots \\ \partial_2 \downarrow & & \partial_2 \downarrow & & \partial_2 \downarrow & & \\ C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \cdots \\ \partial_1 \downarrow & & \partial_1 \downarrow & & \partial_1 \downarrow & & \\ C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \cdots \end{array}$$

*p-persistent $k^{th}$ homology group:*

$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y

$\partial_3 \downarrow$
$C_2 = \mathbf{Z_2}[abc, acd]$
$\partial_2 \downarrow$
$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$
$\partial_1 \downarrow$
$C_0 = \mathbf{Z_2}[a, b, c, d]$
$\partial_0 \downarrow$
0




$\partial_3 \downarrow$
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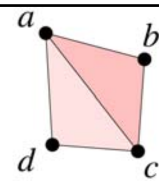
$\partial_0 \downarrow$
0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$

$\partial_3 \downarrow$
$C_2 = \mathbf{Z_2}[abc, acd]$
$\partial_2 \downarrow$
$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$
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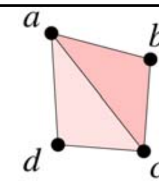
$C_0 = \mathbf{Z_2}[a, b, c, d]$
$\partial_0 \downarrow$
0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}$[abc, acd]

$\partial_2 \downarrow$

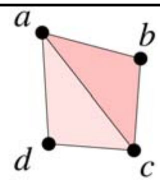
$C_1 = \mathbf{Z_2}$[ab, bc, ac, ad, cd]

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}$[a, b, c, d]

$\partial_0 \downarrow$

0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

a + b + bc + abc + acd

---

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}$[abc, acd]

$\partial_2 \downarrow$

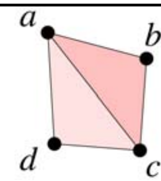
$C_1 = \mathbf{Z_2}$[ab, bc, ac, ad, cd]

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}$[a, b, c, d]

$\partial_0 \downarrow$

0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

a + b $\oplus$ bc $\oplus$ abc + acd

(a + b, bc, abc + acd)

---

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}$[abc, acd]

$\partial_2 \downarrow$

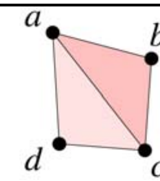
$C_1 = \mathbf{Z_2}$[ab, bc, ac, ad, cd]

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}$[a, b, c, d]

$\partial_0 \downarrow$

0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$

$\partial: \oplus C_k \rightarrow \oplus C_k$

$\partial$(0, ab, 0) = (a + b, 0, 0)

$\partial^2 = 0$

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$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}$[abc, acd]

$\partial_2 \downarrow$

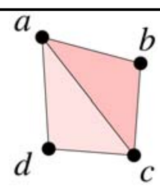
$C_1 = \mathbf{Z_2}$[ab, bc, ac, ad, cd]

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}$[a, b, c, d]

$\partial_0 \downarrow$

0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$

$\partial: \oplus C_k \rightarrow \oplus C_k$

$\partial$(0, ab, 0) = (a + b, 0, 0)

$\partial$(0 $\oplus$ ab $\oplus$ 0) = a+b $\oplus$ 0 $\oplus$ 0

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$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}$[abc, acd]

$\partial_2 \downarrow$

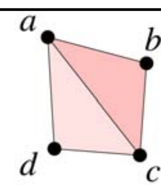
$C_1 = \mathbf{Z_2}$[ab, bc, ac, ad, cd]

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}$[a, b, c, d]

$\partial_0 \downarrow$

0




$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$

$\partial: \oplus C_k \rightarrow \oplus C_k$

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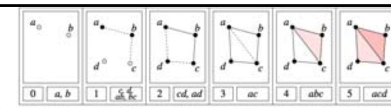


$$C_2^0 \xrightarrow{f^0} C_2^1 \xrightarrow{f^1} C_2^2 \xrightarrow{f^2} \cdots$$
$$\partial_2 \downarrow \quad \partial_2 \downarrow \quad \partial_2 \downarrow$$
$$C_1^0 \xrightarrow{f^0} C_1^1 \xrightarrow{f^1} C_1^2 \xrightarrow{f^2} \cdots$$
$$\partial_1 \downarrow \quad \partial_1 \downarrow \quad \partial_1 \downarrow$$
$$C_0^0 \xrightarrow{f^0} C_0^1 \xrightarrow{f^1} C_0^2 \xrightarrow{f^2} \cdots$$

*p-persistent $k^{th}$ homology group:*

$$H_k^{i,p} = Z_k^i / ( B_k^{i+p} \cap Z_k^i )$$

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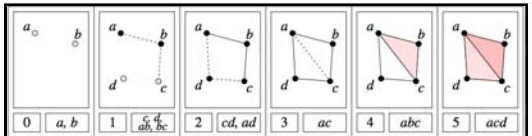
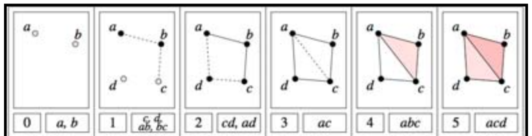
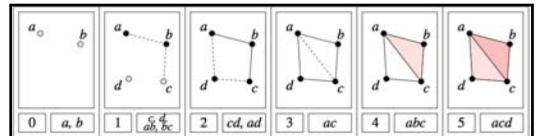



$C_j^0 \to C_j^1 \to C_j^2 \to C_j^3 \to C_j^4 \to C_j^5$



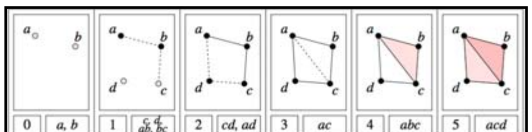

$C_0^0 \to C_0^1 \to C_0^2 \to C_0^3 \to C_0^4 \to C_0^5$




$C_0^0 \to C_0^1 \to C_0^2 \to C_0^3 \to C_0^4 \to C_0^5 \to ..$

$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

$(a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \dots)$




$C_0^0 \to C_0^1 \to C_0^2 \to C_0^3 \to C_0^4 \to C_0^5 \to ..$

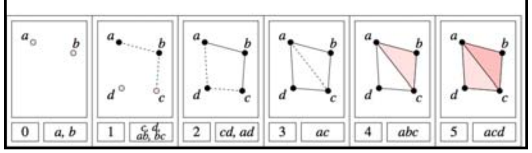
$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

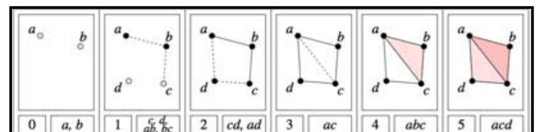
$(a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \dots)$

$t \bullet (a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \dots)$

$= (0, ta, tc + t^2a, t^2c + t^3b, t^3c, t^5b, t^6b, t^7b, \dots)$

$H_0 = \langle a, b, c, d : tc + td, tb + c, ta + tb\rangle$

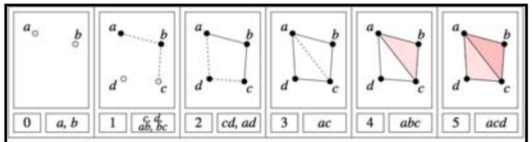



$C_0^0 \to C_0^1 \to C_0^2 \to C_0^3 \to C_0^4 \to C_0^5 \to ..$

$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

a: $\mathbf{Z_2}[t]$ b: $\mathbf{Z_2}[t]$ c: $\Sigma^1 \mathbf{Z_2}[t]$ d: $\Sigma^1 \mathbf{Z_2}[t]$

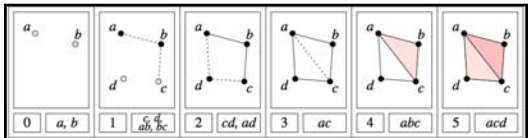



$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$

$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

a: $\mathbf{Z}_2[t]$ b: $\mathbf{Z}_2[t]$ c: $\Sigma^1 \mathbf{Z}_2[t]$ d: $\Sigma^1 \mathbf{Z}_2[t]$

acd: $\Sigma^5 \mathbf{Z}_2[t]$

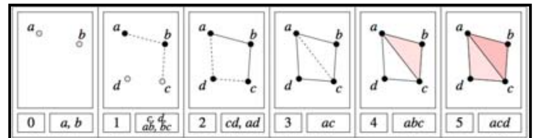


$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$

$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

a: $\mathbf{Z}_2[t] = \{n_0 + n_1t + n_2t^2 + \dots + n_kt^k : n_i \text{ in } \mathbf{Z}_2, k \text{ in } \mathbf{Z}_+\}$

$= \{ (n_0, n_1, n_2, \dots, n_k, 0, 0, \dots) : n_i \text{ in } \mathbf{Z}_2, k \text{ in } \mathbf{Z}_+\}$

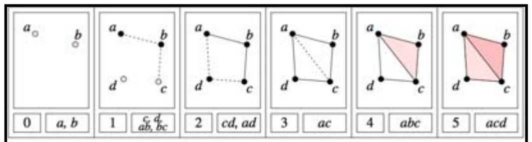


$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$

$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

b: $\mathbf{Z}_2[t] = \{n_0 + n_1t + n_2t^2 + \dots + n_kt^k : n_i \text{ in } \mathbf{Z}_2, k \text{ in } \mathbf{Z}_+\}$

$= \{ (n_0, n_1, n_2, \dots, n_k, 0, 0, \dots) : n_i \text{ in } \mathbf{Z}_2, k \text{ in } \mathbf{Z}_+\}$

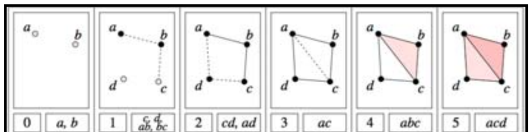


$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$

$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

c: $\Sigma^1\mathbf{Z}_2[t] = \{n_0t + n_1t^2 + \dots : n_i \text{ in } \mathbf{Z}_2\}$

$= \{ (0, n_0, n_1, \dots) : n_i \text{ in } \mathbf{Z}_2\}$



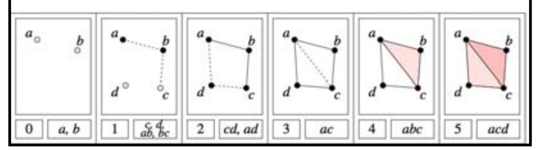
$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$

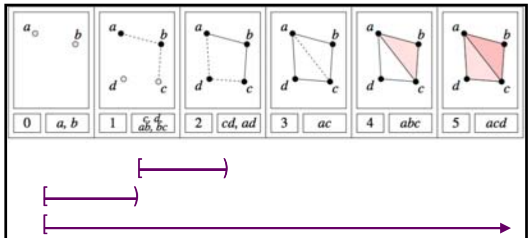
$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \dots$

d: $\Sigma^1\mathbf{Z}_2[t] = \{n_0t + n_1t^2 + \dots : n_i \text{ in } \mathbf{Z}_2\}$

$= \{ (0, n_0, n_1, \dots) : n_i \text{ in } \mathbf{Z}_2\}$

$H_0 = \langle a, b, c, d : tc + td, tb + c, ta + tb\rangle$

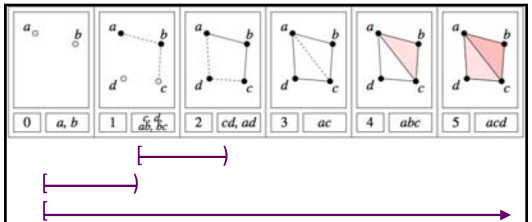





$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

a: $\mathbf{Z_2}[t] = \{n_0 + n_1t + n_2t^2 + \ldots + n_kt^k :$ $n_i$ in $\mathbf{Z_2}$, k in $\mathbf{Z_+}$}

= { $(n_0, n_1, n_2, \ldots, n_k, 0, 0, \ldots)$ : $n_i$ in $\mathbf{Z_2}$, k in $\mathbf{Z_+}$}

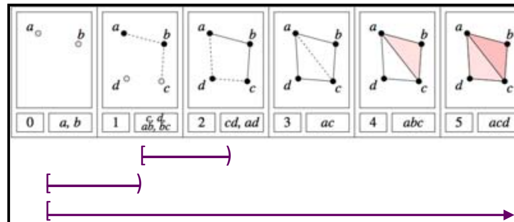



$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

$\mathbf{Z_2}[t] = \{n_0 + n_1t + n_2t^2 + \ldots :$ $n_i$ in $\mathbf{Z_2}$}

b: $\mathbf{Z_2}[t]/t = \{n_0 :$ $n_0$ in $\mathbf{Z_2}\} = \mathbf{Z_2}$

= { $(n_0, 0, 0, 0, \ldots)$ : $n_i$ in $\mathbf{Z_2}$}

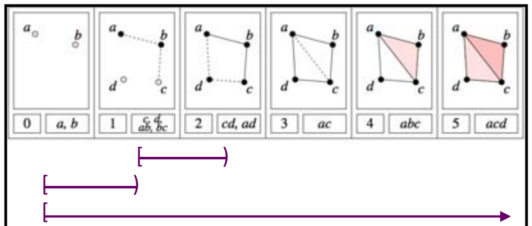



$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

$\mathbf{Z_2}[t] = \{n_0 + n_1t + n_2t^2 + \ldots :$ $n_i$ in $\mathbf{Z_2}$}

d: $\Sigma^1 \mathbf{Z_2}[t]/t = \Sigma^1 \{n_0 :$ $n_0$ in $\mathbf{Z_2}$}

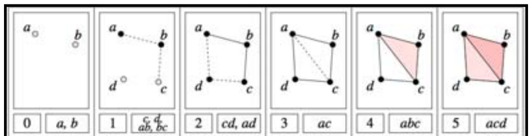
= { $(0, n_0, 0, 0, 0, \ldots)$ : $n_0$ in $\mathbf{Z_2}$}




$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

$\mathbf{Z_2}[t] = \{n_0 + n_1t + n_2t^2 + \ldots :$ $n_i$ in $\mathbf{Z_2}$}

c: $\Sigma^1 \mathbf{Z_2}[t]/1$ = { (0, 0, 0, … ) }




$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \ldots$

$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

a: $\mathbf{Z_2}[t] = \{n_0 + n_1t + n_2t^2 + \ldots\ n_jt^j :$ $n_i$ in $\mathbf{Z_2}$}

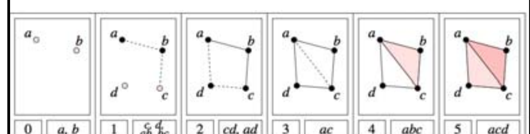
b: $\mathbf{Z_2}[t]/t = \{n_0 :$ $n_0$ in $\mathbf{Z_2}\} = \mathbf{Z_2}$

c: $\Sigma^1 \mathbf{Z_2}[t]/1$ = empty set

d: $\Sigma^1 \mathbf{Z_2}[t]/t = \{n_0 :$ $n_0$ in $\mathbf{Z_2}\} = \{(0, n_0, 0, 0, 0, \ldots)\}$

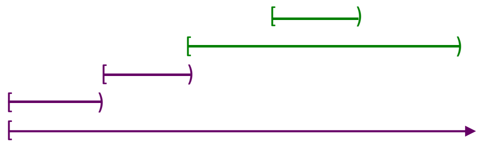
$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

$= \mathbf{Z_2}[t] \oplus \mathbf{Z_2}[t]/t \oplus (\Sigma^1 \mathbf{Z_2}[t])/t$

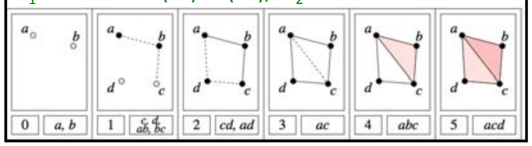


$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

$H_1$ = < $z_1$, $z_2$ : t $z_2$, $t^3 z_1 + t^2 z_2$ >



$z_1$ = ad + cd + t(bc) + t(ab), $z_2$ = ac + $t^2$bc + $t^2$ab



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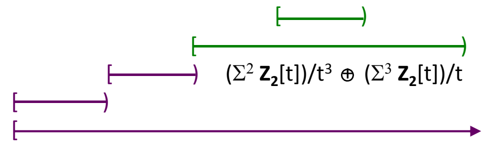
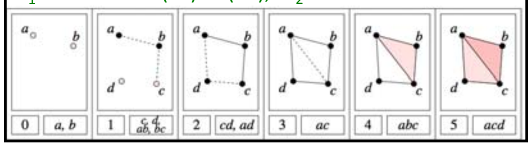
$H_0$ = < a, b, c, d : tc + td, tb + c, ta + tb>

$H_1$ = < $z_1$, $z_2$ : t $z_2$, $t^3 z_1 + t^2 z_2$ >



$(\Sigma^2 \mathbf{Z_2}[t])/t^3 \oplus (\Sigma^3 \mathbf{Z_2}[t])/t$

$z_1$ = ad + cd + t(bc) + t(ab), $z_2$ = ac + $t^2$bc + $t^2$ab



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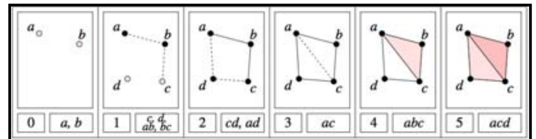


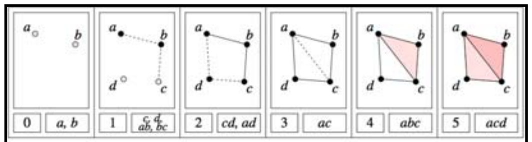
$H_1$ = < $z_1$, $z_2$ : t $z_2$, $t^3 z_1 + t^2 z_2$ >

$\mathbf{Z_2}[t]$ = $\{n_0 + n_1 t + n_2 t^2 + n_2 t^3 + n_2 t^4 + \ldots \ : n_i \text{ in } \mathbf{Z_2}\}$

$z_1$: $(\Sigma^2 \mathbf{Z_2}[t])/t^3$

= { (0, 0, $n_0$, $n_1$, $n_2$, 0, 0, 0,… ) : $n_i$ in $\mathbf{Z_2}$}

---



$H_1$ = < $z_1$, $z_2$ : t $z_2$, $t^3 z_1 + t^2 z_2$ >

$\mathbf{Z_2}[t]$ = $\{n_0 + n_1 t + n_2 t^2 + n_2 t^3 + n_2 t^4 + \ldots \ : n_i \text{ in } \mathbf{Z_2}\}$

$z_2$: $(\Sigma^3 \mathbf{Z_2}[t])/t$

= { (0, 0, 0, $n_0$, 0, 0, 0, … ) : $n_i$ in $\mathbf{Z_2}$}

---

In general when calculating homology over the field **F**

$$H_k = \left( \bigoplus_{i=1}^{n} \Sigma^{\alpha_i} \mathbf{F}[t] \right) \oplus \left( \bigoplus_{i=1}^{n} \Sigma^{\gamma_j} \mathbf{F}[t]/(t^{k_j}) \right)$$

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MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

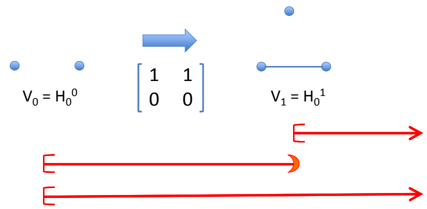
Sept 30, 2013: Multidimensional Persistence.

Fall 2013 course offered through the University of Iowa Division of Continuing Education

Isabel K. Darcy, Department of Mathematics
Applied Mathematical and Computational Sciences, University of Iowa

http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

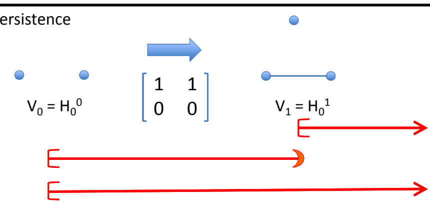
1D persistence




Rank L(0, 0) = 2; Rank L(0, 1) = 1; Rank L(1, 1) = 2

Note Rank determines persistence bars and vice versa

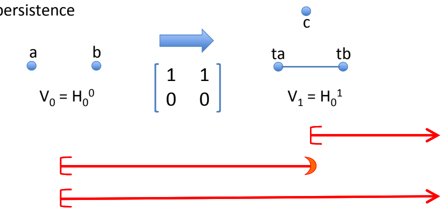
1D persistence




Rank L(0, 0) = 2; $H_0^{0,0}$ Rank L(0, 1) = 1; $H_0^{0,1}$ Rank L(1, 1) = 2 $H_0^{1,0}$

$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i) = L(i, i+p)(H_k^i)$

1D persistence




$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} ab \\ \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} \end{array}$$

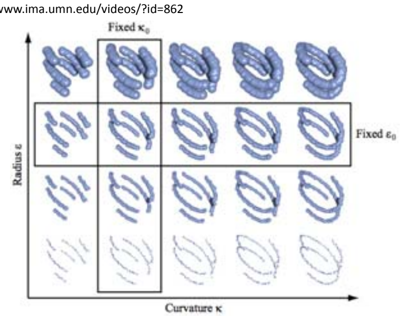
$\partial(ab) = t(b - a)$

$H_0$ Persistence module: $F[t] \oplus F[t]/(t) \oplus \Sigma^1 F[t]$

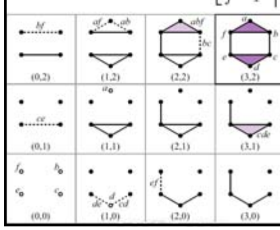
$(\Sigma f_n t^n, \quad f, \quad t \, \Sigma f_n t^n)$

The Theory of Multidimensional Persistence, Gunnar Carlsson, Afra Zomorodian

**"Persistence and Point Clouds" Functoriality, diagrams, difficulties in classifying diagrams, multidimensional persistence, Gröbner bases,** Gunnar Carlsson
http://www.ima.umn.edu/videos/?id=862




| | | ab | bc | cd | de | ef | af | bf | ce |
|---|---|---|---|---|---|---|---|---|---|
| | | $x_1x_2^2$ | $x_1^2x_2^2$ | $x_1$ | $x_1$ | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | $x_2$ |
| a | $x_1x_2$ | $x_2$ | 0 | 0 | 0 | 0 | $x_2$ | 0 | 0 |
| d | $x_1$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| b | 1 | $x_1x_2^2$ | $x_1^2x_2^2$ | 0 | 0 | 0 | 0 | $x_2^2$ | 0 |
| c | 1 | 0 | $x_1^2x_2^2$ | $x_1$ | 0 | 0 | 0 | 0 | $x_2$ |
| e | 1 | 0 | 0 | 0 | $x_1$ | $x_1^2$ | 0 | 0 | $x_2$ |
| f | 1 | 0 | 0 | 0 | 0 | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | 0 |



Computing Multidimensional Persistence,
Gunnar Carlsson, Gurjeet Singh, and Afra Zomorodian

| | ab | bc | cd | de | ef | af | bf | ce |
|---|---|---|---|---|---|---|---|---|
| a | $x_2$ | 0 | 0 | 0 | 0 | $x_2$ | 0 | 0 |
| d | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| b | $x_1x_2^2$ | $x_1^2x_2^2$ | 0 | 0 | 0 | 0 | $x_2^2$ | 0 |
| c | 0 | $x_1^2x_2^2$ | $x_1$ | 0 | 0 | 0 | 0 | $x_2$ |
| e | 0 | 0 | 0 | $x_1$ | $x_1^2$ | 0 | 0 | $x_2$ |
| f | 0 | 0 | 0 | 0 | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | 0 |

| | ab | bc | cd | de | ef | af + ab | bf | ce |
|---|---|---|---|---|---|---|---|---|
| a | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| d | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| b | $x_1x_2^2$ | $x_1^2x_2^2$ | 0 | 0 | 0 | $x_1x_2^2$ | $x_2^2$ | 0 |
| c | 0 | $x_1^2x_2^2$ | $x_1$ | 0 | 0 | 0 | 0 | $x_2$ |
| e | 0 | 0 | 0 | $x_1$ | $x_1^2$ | 0 | 0 | $x_2$ |
| f | 0 | 0 | 0 | 0 | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | 0 |

$$\begin{array}{c|cccccccc} & ab & bc & cd & de & ef & af+ab & bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\ c & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\ f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & bc & cd & de+cd & ef & af+ab & bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\ c & 0 & x_1^2x_2^2 & x_1 & x_1 & 0 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\ f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & bc & cd & de+cd & ef & af+ab & bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & x_1^2x_2^2 & 0 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\ c & 0 & x_1^2x_2^2 & x_1 & x_1 & 0 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\ f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bc & de+cd & ef & af+ab & bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_1^2x_2^2 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\ c & 0 & x_1 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\ f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bc & de+cd & ef & af+ab & bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_1^2x_2^2 & 0 & 0 & x_1x_2^2 & x_2^2 & 0 \\ c & 0 & x_1 & x_1^2x_2^2 & x_1 & 0 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & x_1 & x_1^2 & 0 & 0 & x_2 \\ f & 0 & 0 & 0 & 0 & x_1^2 & x_1x_2^2 & x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & bc & de+cd & ef & af+ab & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & x_1^2x_2^2 & 0 & 0 & x_1x_2^2 & 0 \\ c & 0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & x_1x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & bc & de+cd & ef & af+ab & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & x_1^2x_2^2 & 0 & 0 & x_1x_2^2 & 0 \\ c & 0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & x_1x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & bc+x_1^2bf & de+cd & ef & af+ab & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & x_1x_2^2 & 0 \\ c & 0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & x_1^2x_2^2 & 0 & x_1^2 & x_1x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & bc+x_1^2bf & de+cd & ef & af+ab & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & x_1x_2^2 & 0 \\ c & 0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & x_1^2x_2^2 & 0 & x_1^2 & x_1x_2^2 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & bc+x_1^2bf & de+cd & ef & af+ab+x_1bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & x_1^2x_2^2 & 0 & x_1^2 & 0 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & bc+x_1^2bf & de+cd & ef & af+ab+x_1bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & x_1 & 0 & x_1^2x_2^2 & x_1 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & 0 & x_1 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & x_1^2x_2^2 & 0 & x_1^2 & 0 & 0 \end{array}$$

$$\begin{array}{c|cccccccc} & ab & cd & bf & de+cd & bc+x_1^2bf & ef & af+ab+x_1bf & ce \\ \hline a & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & x_1 & 0 & x_1 & x_1^2x_2^2 & 0 & 0 & x_2 \\ e & 0 & 0 & 0 & x_1 & 0 & x_1^2 & 0 & x_2 \\ f & 0 & 0 & x_2^2 & 0 & x_1^2x_2^2 & x_1^2 & 0 & 0 \end{array}$$

| | $ab$ | $cd$ | $bf$ | $de+cd$ | $bc+x_1^2bf$ | $ef$ | $af+ab+x_1bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $x_1$ | 0 | $x_1$ | $x_1^2x_2^2$ | 0 | 0 | $x_2$ |
| $e$ | 0 | 0 | 0 | $x_1$ | 0 | $x_1^2$ | 0 | $x_2$ |
| $f$ | 0 | 0 | $x_2^2$ | 0 | $x_1^2x_2^2$ | $x_1^2$ | 0 | 0 |

| | $ab$ | $cd$ | $bf$ | $de+cd$ | $bc+x_1^2bf+x_1x_2^2(de+cd)$ | $ef$ | $af+ab+x_1bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $x_1$ | 0 | $x_1$ | 0 | 0 | 0 | $x_2$ |
| $e$ | 0 | 0 | 0 | $x_1$ | $x_1^2x_2^2$ | $x_1^2$ | 0 | $x_2$ |
| $f$ | 0 | 0 | $x_2^2$ | 0 | $x_1^2x_2^2$ | $x_1^2$ | 0 | 0 |

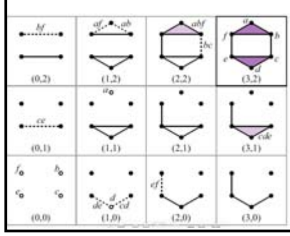
| | $ab$ | $cd$ | $bf$ | $de+cd$ | $bc+x_1^2bf+x_1x_2^2(de+cd)$ | $ef$ | $af+ab+x_1bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $x_1$ | 0 | $x_1$ | 0 | 0 | 0 | $x_2$ |
| $e$ | 0 | 0 | 0 | $x_1$ | $x_1^2x_2^2$ | $x_1^2$ | 0 | $x_2$ |
| $f$ | 0 | 0 | $x_2^2$ | 0 | $x_1^2x_2^2$ | $x_1^2$ | 0 | 0 |

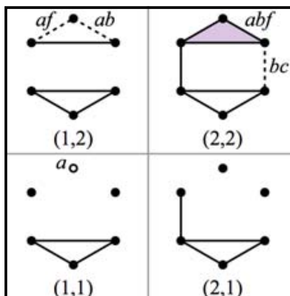
| | $ab$ | $cd$ | $bf$ | $de+cd$ | $ef$ | $bc+x_1^2bf+x_1x_2^2(de+cd)$ | $af+ab+x_1bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $x_1$ | 0 | $x_1$ | 0 | 0 | 0 | $x_2$ |
| $e$ | 0 | 0 | 0 | $x_1$ | $x_1^2$ | $x_1^2x_2^2$ | 0 | $x_2$ |
| $f$ | 0 | 0 | $x_2^2$ | 0 | $x_1^2$ | $x_1^2x_2^2$ | 0 | 0 |

| | $ab$ | $cd$ | $bf$ | $de+cd$ | $ef$ | $bc+x_1^2bf+x_1x_2^2(de+cd)$ | $af+ab+x_1bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $x_1$ | 0 | $x_1$ | 0 | 0 | 0 | $x_2$ |
| $e$ | 0 | 0 | 0 | $x_1$ | $x_1^2$ | $x_1^2x_2^2$ | 0 | $x_2$ |
| $f$ | 0 | 0 | $x_2^2$ | 0 | $x_1^2$ | $x_1^2x_2^2$ | 0 | 0 |

| | $ab$ | $cd$ | $bf$ | $de+cd$ | $ef$ | $bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef$ | $af+ab+x_1bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $x_1$ | 0 | $x_1$ | 0 | 0 | 0 | $x_2$ |
| $e$ | 0 | 0 | 0 | $x_1$ | $x_1^2$ | 0 | 0 | $x_2$ |
| $f$ | 0 | 0 | $x_2^2$ | 0 | $x_1^2$ | 0 | 0 | 0 |

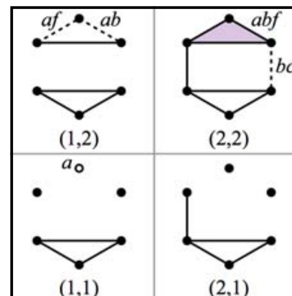
| | $ab$ | $cd$ | $bf$ | $de+cd$ | $ce$ | $ef$ | $bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef$ | $af+ab+x_1bf$ |
|---|---|---|---|---|---|---|---|---|
| $a^{(1,1)}$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d^{(1,0)}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b^{(0,0)}$ | $x_1x_2^2$ | 0 | $x_2^2$ | 0 | 0 | 0 | 0 | 0 |
| $c^{(0,0)}$ | 0 | $x_1$ | 0 | $x_1$ | $x_2$ | 0 | 0 | 0 |
| $e^{(0,0)}$ | 0 | 0 | 0 | $x_1$ | $x_2$ | $x_1^2$ | 0 | 0 |
| $f^{(0,0)}$ | 0 | 0 | $x_2^2$ | 0 | 0 | $x_1^2$ | 0 | 0 |




$ab$

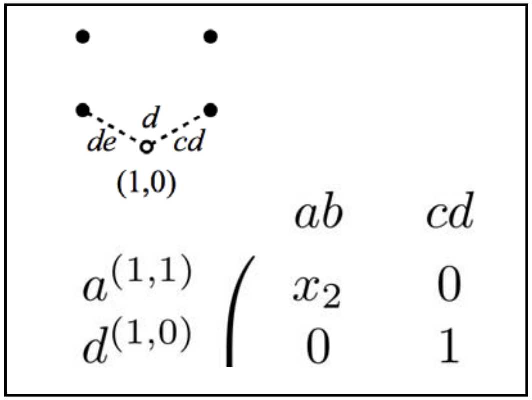
$a^{(1,1)}$ / $x_2$

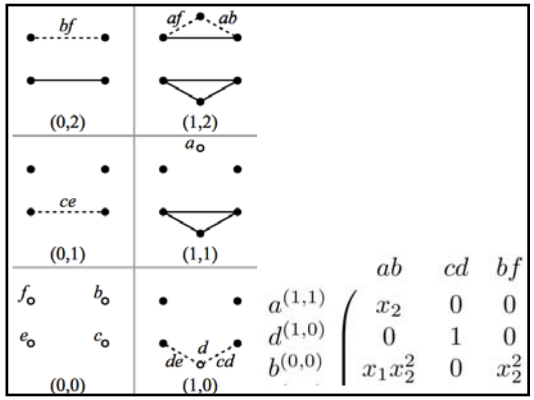


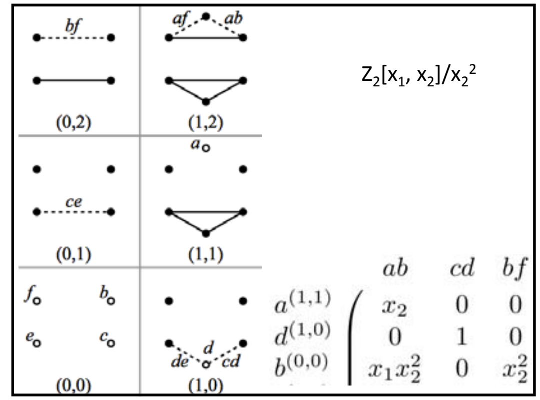

$\Sigma^{(1,1)} Z_2[x_1, x_2]/x_2$

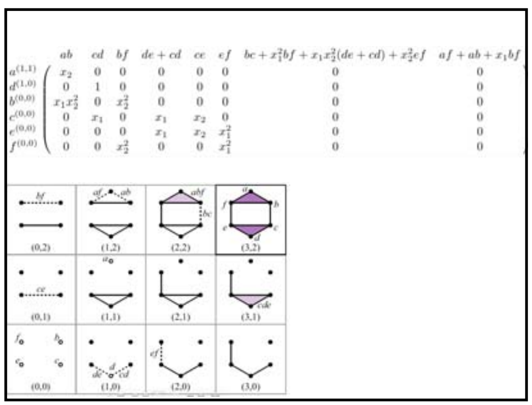
$ab$

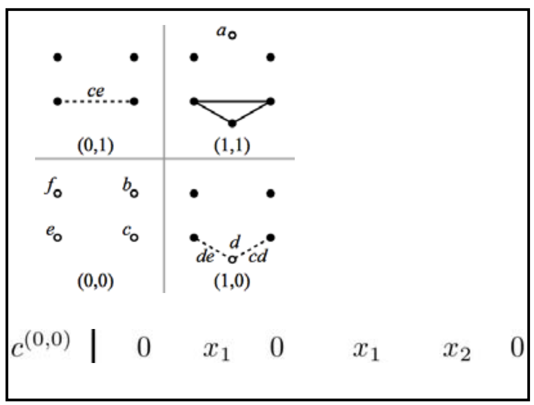
$a^{(1,1)}$ / $x_2$

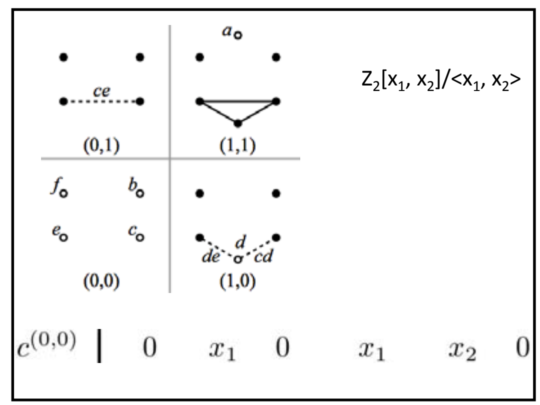

d
de
cd
(1,0)
ab cd
a^(1,1) x_2 0
d^(1,0) 0 1

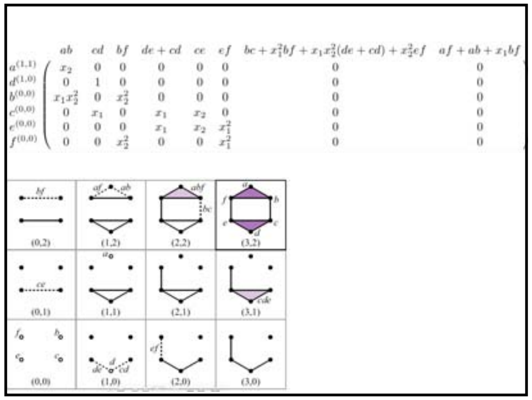

bf
af ab
(0,2)
(1,2)
a
ce
(0,1)
(1,1)
f b
e c
d
de cd
(0,0)
(1,0)
ab cd bf
a^(1,1) x_2 0 0
d^(1,0) 0 1 0
b^(0,0) x_1x_2^2 0 x_2^2


bf
af ab
(0,2)
(1,2)
a
ce
(0,1)
(1,1)
f b
e c
d
de cd
(0,0)
(1,0)
Z_2[x_1, x_2]/x_2^2
ab cd bf
a^(1,1) x_2 0 0
d^(1,0) 0 1 0
b^(0,0) x_1x_2^2 0 x_2^2

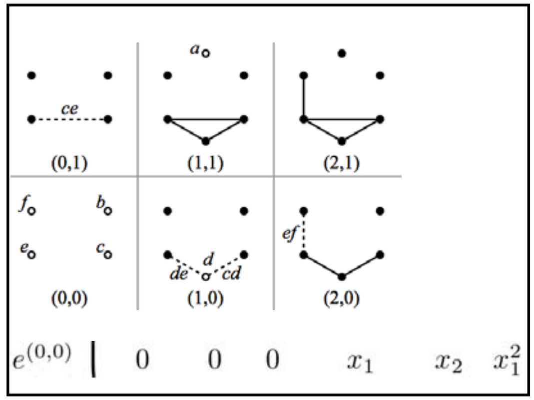

ab cd bf de+cd ce ef bc+x_1^2bf+x_1x_2^2(de+cd)+x_2^2ef af+ab+x_1bf
a^(1,1) x_2 0 0 0 0 0 0 0
d^(1,0) 0 1 0 0 0 0 0 0
b^(0,0) x_1x_2^2 0 x_2^2 0 0 0 0 0
c^(0,0) 0 x_1 0 x_1 x_2 0 0 0
e^(0,0) 0 0 0 x_1 x_2 x_1^2 0 0
f^(0,0) 0 0 x_2^2 0 0 x_1^2 0 0
bf
af ab
abf
bc
(0,2)
(1,2)
(2,2)
(3,2)
(0,1)
(1,1)
(2,1)
(3,1)
(0,0)
(1,0)
(2,0)
(3,0)


a
ce
(0,1)
(1,1)
f b
e c
d
de cd
(0,0)
(1,0)
c^(0,0) 0 x_1 0 x_1 x_2 0

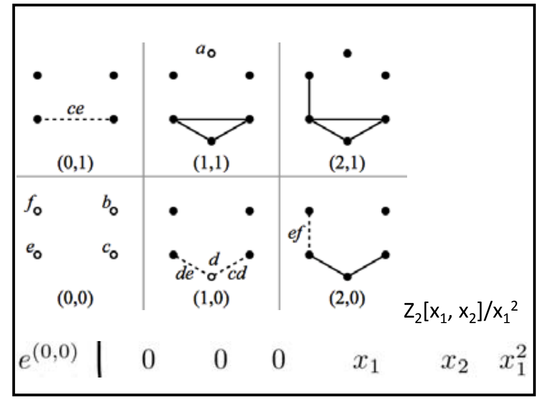

a
ce
(0,1)
(1,1)
Z_2[x_1, x_2]/<x_1, x_2>
f b
e c
d
de cd
(0,0)
(1,0)
c^(0,0) 0 x_1 0 x_1 x_2 0



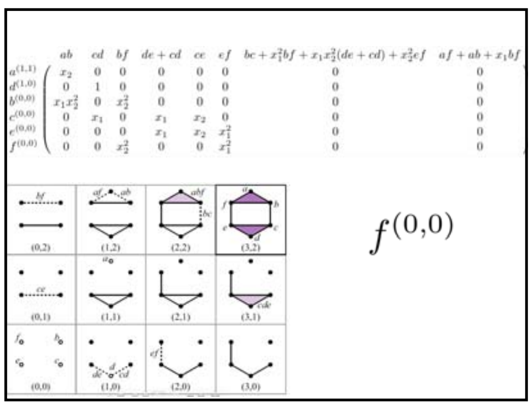
ab cd bf de + cd ce ef $bc + x_1^2bf + x_1x_2^2(de+cd) + x_2^2ef$ $af + ab + x_1bf$
$a^{(1,1)}$ $x_2$ 0 0 0 0 0 0 0
$d^{(1,0)}$ 0 1 0 0 0 0 0 0
$b^{(0,0)}$ $x_1x_2^2$ 0 $x_2^2$ 0 0 0 0 0
$c^{(0,0)}$ 0 $x_1$ 0 $x_1$ $x_2$ 0 0 0
$e^{(0,0)}$ 0 0 0 $x_1$ $x_2$ $x_1^2$ 0 0
$f^{(0,0)}$ 0 0 $x_2^2$ 0 0 $x_1^2$ 0 0

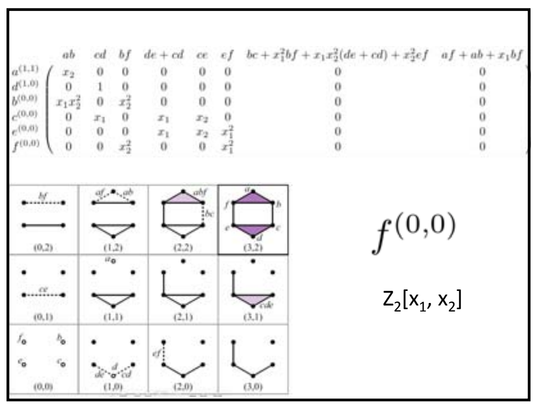
ce
a
(0,1)
(1,1)
(2,1)
f b
e c
d
de cd
ef
(0,0)
(1,0)
(2,0)
$e^{(0,0)}$ | 0 0 0 $x_1$ $x_2$ $x_1^2$

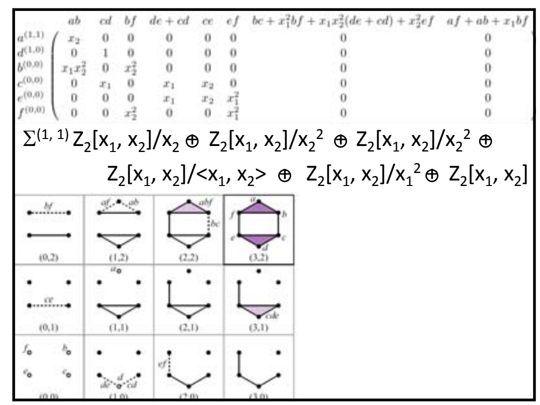
ce
a
(0,1)
(1,1)
(2,1)
f b
e c
d
de cd
ef
(0,0)
(1,0)
(2,0)
$Z_2[x_1, x_2]/x_1^2$
$e^{(0,0)}$ | 0 0 0 $x_1$ $x_2$ $x_1^2$

ab cd bf de + cd ce ef $bc + x_1^2bf + x_1x_2^2(de+cd) + x_2^2ef$ $af + ab + x_1bf$
$a^{(1,1)}$ $x_2$ 0 0 0 0 0 0 0
$d^{(1,0)}$ 0 1 0 0 0 0 0 0
$b^{(0,0)}$ $x_1x_2^2$ 0 $x_2^2$ 0 0 0 0 0
$c^{(0,0)}$ 0 $x_1$ 0 $x_1$ $x_2$ 0 0 0
$e^{(0,0)}$ 0 0 0 $x_1$ $x_2$ $x_1^2$ 0 0
$f^{(0,0)}$ 0 0 $x_2^2$ 0 0 $x_1^2$ 0 0
$f^{(0,0)}$

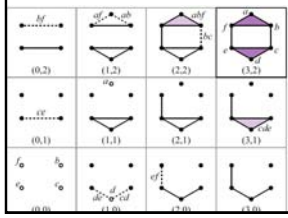
ab cd bf de + cd ce ef $bc + x_1^2bf + x_1x_2^2(de+cd) + x_2^2ef$ $af + ab + x_1bf$
$a^{(1,1)}$ $x_2$ 0 0 0 0 0 0 0
$d^{(1,0)}$ 0 1 0 0 0 0 0 0
$b^{(0,0)}$ $x_1x_2^2$ 0 $x_2^2$ 0 0 0 0 0
$c^{(0,0)}$ 0 $x_1$ 0 $x_1$ $x_2$ 0 0 0
$e^{(0,0)}$ 0 0 0 $x_1$ $x_2$ $x_1^2$ 0 0
$f^{(0,0)}$ 0 0 $x_2^2$ 0 0 $x_1^2$ 0 0
$f^{(0,0)}$
$Z_2[x_1, x_2]$

ab cd bf de + cd ce ef $bc + x_1^2bf + x_1x_2^2(de+cd) + x_2^2ef$ $af + ab + x_1bf$
$a^{(1,1)}$ $x_2$ 0 0 0 0 0 0 0
$d^{(1,0)}$ 0 1 0 0 0 0 0 0
$b^{(0,0)}$ $x_1x_2^2$ 0 $x_2^2$ 0 0 0 0 0
$c^{(0,0)}$ 0 $x_1$ 0 $x_1$ $x_2$ 0 0 0
$e^{(0,0)}$ 0 0 0 $x_1$ $x_2$ $x_1^2$ 0 0
$f^{(0,0)}$ 0 0 $x_2^2$ 0 0 $x_1^2$ 0 0
$\Sigma^{(1,1)} Z_2[x_1, x_2]/x_2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus$
$Z_2[x_1, x_2]/\langle x_1, x_2\rangle \oplus Z_2[x_1, x_2]/x_1^2 \oplus Z_2[x_1, x_2]$

$$
\begin{array}{c|cccccccc}
 & ab & cd & bf & de+cd & ce & ef & bc + x_1^2 bf + x_1 x_2^2(de+cd) + x_2^2 ef & af + ab + x_1 bf \\
\hline
a^{(1,1)} & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d^{(1,0)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b^{(0,0)} & x_1 x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
c^{(0,0)} & 0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
e^{(0,0)} & 0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
f^{(0,0)} & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & 0
\end{array}
$$

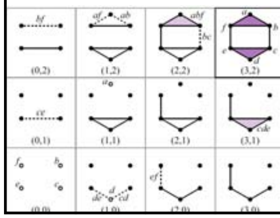
$\Sigma^{(1,1)} Z_2[x_1, x_2]/x_2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus$
$Z_2[x_1, x_2]/\langle x_1, x_2\rangle \oplus Z_2[x_1, x_2]/x_1^2 \oplus Z_2[x_1, x_2]$



In general, use Buchberger's/ algorithm for constructing a Grobner basis.
Algorithm described in Computing Multidimensional Persistence, by Carlsson, Singh, Zomorodian, can be computed in polynomial time.

$$
\begin{array}{c|cccccccc}
 & ab & cd & bf & de+cd & ce & ef & bc + x_1^2 bf + x_1 x_2^2(de+cd) + x_2^2 ef & af + ab + x_1 bf \\
\hline
a^{(1,1)} & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d^{(1,0)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b^{(0,0)} & x_1 x_2^2 & 0 & x_2^2 & 0 & 0 & 0 & 0 & 0 \\
c^{(0,0)} & 0 & x_1 & 0 & x_1 & x_2 & 0 & 0 & 0 \\
e^{(0,0)} & 0 & 0 & 0 & x_1 & x_2 & x_1^2 & 0 & 0 \\
f^{(0,0)} & 0 & 0 & x_2^2 & 0 & 0 & x_1^2 & 0 & 0
\end{array}
$$

$\Sigma^{(1,1)} Z_2[x_1, x_2]/x_2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus Z_2[x_1, x_2]/x_2^2 \oplus$
$Z_2[x_1, x_2]/\langle x_1, x_2\rangle \oplus Z_2[x_1, x_2]/x_1^2 \oplus Z_2[x_1, x_2]$



In general, use Dionysus software,

http://www.mrzv.org/software/dionysus/

See next lecture, 1 week from today.

MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Oct 16, 2013: Zigzag Persistence and installing Dionysus part I.

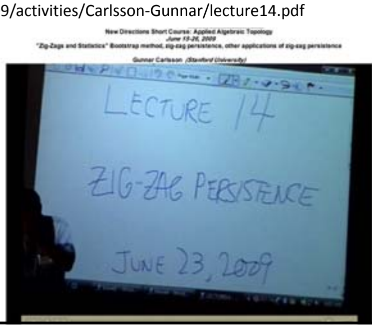
Fall 2013 course offered through the University of Iowa Division of Continuing Education

Isabel K. Darcy, Department of Mathematics Applied Mathematical and Computational Sciences, University of Iowa

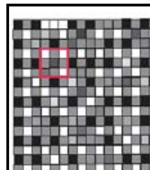
http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

http://www.ima.umn.edu/videos/?id=863

http://www.ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-Gunnar/lecture14.pdf




http://geometrica.saclay.inria.fr/workshops/TGDA_07_2009/workshop_files/slides/deSilva_TGDA.pdf

**Observations:**

1. Each patch gives a vector in $\mathbb{R}^9$
2. Most patches will be nearly constant, or *low contrast*, because of the presence of regions of solid shading in most images

3. Low contrast will dominate statistics, not interesting

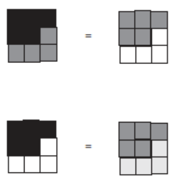
Lee-Mumford-Pedersen [LMP] study only high contrast patches.

Collection: $4.5 \times 10^6$ high contrast patches from a collection of images obtained by van Hateren and van der Schaaf

Recall from Sept 20 lecture

- Normalize mean intensity by subtracting mean from each pixel value to obtain patches with mean intensity $= 0$
- Puts data on an 8-dimensional hyperplane, $\cong \mathbb{R}^8$
- Means that we will consider as equivalent patches which can be obtained from each other by turning the intensity knob

- Normalize contrast by dividing by the $D$-norm, so obtain patches with $D$-norm $= 1$
- Means that data now lies on a 7-D ellipsoid, $\cong S^7$



## Codensity

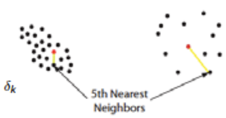
For integer $k > 0$, and PCD $\mathbb{X}$

$$\delta_k(x) = d(x, x')$$

$$x' = \text{ any } k\text{-th nearest neighbor to } x \in \mathbb{X}$$

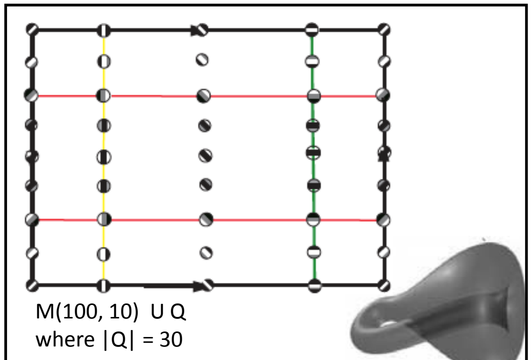
$$\delta_k(x) \text{ large} \Longrightarrow x \text{ is sparse}$$

$$\delta_k(x) \text{ small} \Longrightarrow x \text{ is dense}$$



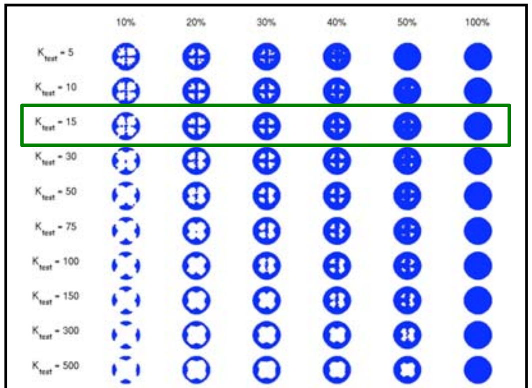

$\mathcal{M}[k, T]$ is $T\%$ densest points as measured by $\delta_k$

What is the persistent homology of these $\mathcal{M}[k, T]$?



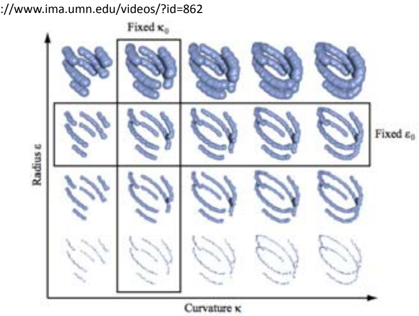
M(100, 10) U Q
where |Q| = 30

On the Local Behavior of Spaces of Natural Images, Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, Afra Zomorodian, International Journal of Computer Vision 2008, pp 1-12.

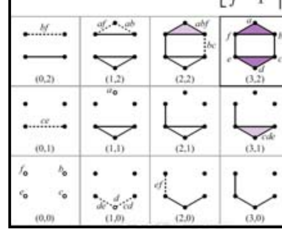



The Theory of Multidimensional Persistence, Gunnar Carlsson, Afra Zomorodian

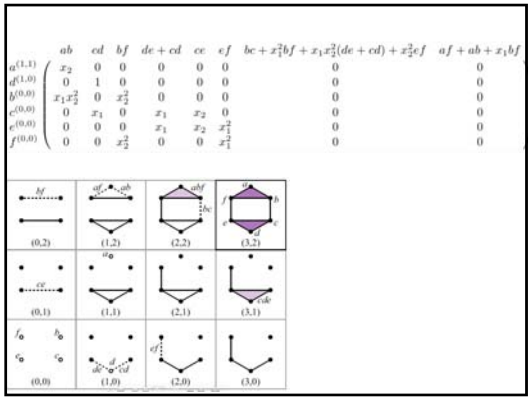
**"Persistence and Point Clouds" Functoriality, diagrams, difficulties in classifying diagrams, multidimensional persistence, Gröbner bases,** Gunnar Carlsson
http://www.ima.umn.edu/videos/?id=862

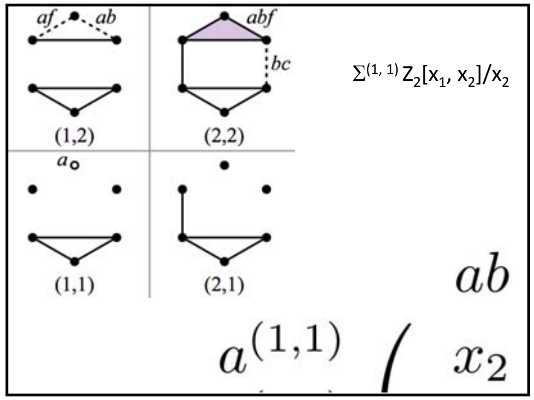


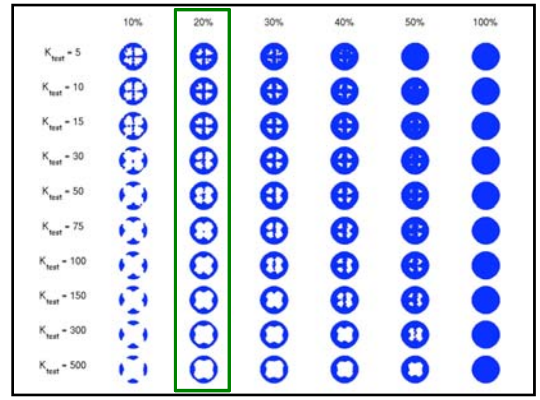

| | | $ab$ | $bc$ | $cd$ | $de$ | $ef$ | $af$ | $bf$ | $ce$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $x_1x_2^2$ | $x_1^2x_2^2$ | $x_1$ | $x_1$ | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | $x_2$ |
| $a$ | $x_1x_2$ | $x_2$ | 0 | 0 | 0 | 0 | $x_2$ | 0 | 0 |
| $d$ | $x_1$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $b$ | 1 | $x_1x_2^2$ | $x_1^2x_2^2$ | 0 | 0 | 0 | 0 | $x_2^2$ | 0 |
| $c$ | 1 | 0 | $x_1^2x_2^2$ | $x_1$ | 0 | 0 | 0 | 0 | $x_2$ |
| $e$ | 1 | 0 | 0 | 0 | $x_1$ | $x_1^2$ | 0 | 0 | $x_2$ |
| $f$ | 1 | 0 | 0 | 0 | 0 | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | 0 |




Computing Multidimensional Persistence, Gunnar Carlsson, Gurjeet Singh, and Afra Zomorodian

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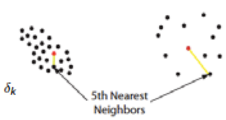


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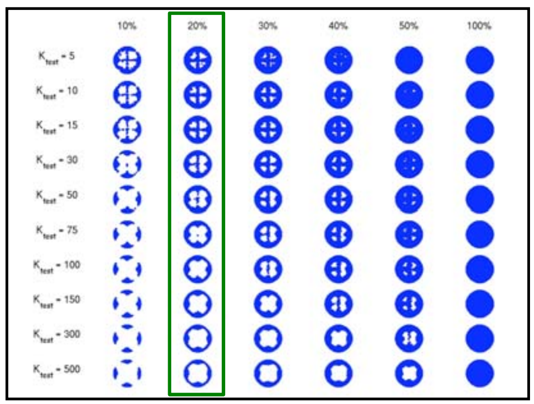
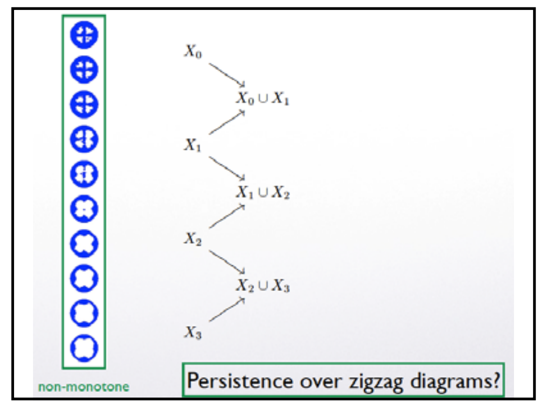
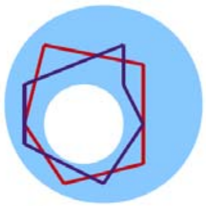


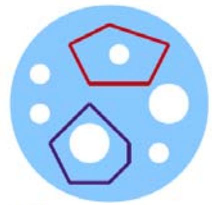
Witness Complexes



$$\mathrm{VR}(X_i, \epsilon) \subset \mathrm{VR}(X_i \cup X_j, \epsilon) \supset \mathrm{VR}(X_j, \epsilon)$$

$$\mathrm{H}_p(\mathrm{VR}(X_i, \epsilon)) \to \mathrm{H}_p(\mathrm{VR}(X_i \cup X_j, \epsilon)) \leftarrow \mathrm{H}_p(\mathrm{VR}(X_j, \epsilon$$

Witness Complexes



$$\mathrm{VR}(X_i, \epsilon) \subset \mathrm{VR}(X_i \cup X_j, \epsilon) \supset \mathrm{VR}(X_j, \epsilon)$$

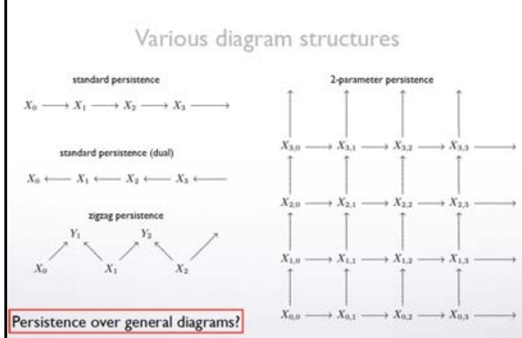
$$\mathrm{H}_p(\mathrm{VR}(X_i, \epsilon)) \to \mathrm{H}_p(\mathrm{VR}(X_i \cup X_j, \epsilon)) \leftarrow \mathrm{H}_p(\mathrm{VR}(X_j, \epsilon$$

Time varying data

$X[t_0, t_1]$ = data points existing at time t for t in $[t_0, t_1]$

$X[t_1, t_2]$ ↙ ↘ $X[t_2, t_3]$ ↙ ↘

$X[t_0, t_2]$ $X[t_1, t_3]$ $X[t_2, t_4]$




Time varying data

$X[t_0, t_1]$ = data points existing at time t for t in $[t_0, t_1]$

$X[t_1, t_2]$ ↙ ↘ $X[t_2, t_3]$ ↙ ↘

$X[t_0, t_2]$ $X[t_1, t_3]$ $X[t_2, t_4]$

$VR(X[t_1, t_2], \varepsilon)$ ↙ ↘ $VR(X[t_2, t_3], \varepsilon)$ ↙ ↘

$VR(X[t_0, t_2], \varepsilon)$ $VR(X[t_1, t_3], \varepsilon)$ $VR(X[t_2, t_4], \varepsilon)$

Time varying data

$X[t_0, t_1]$ = data points existing at time t for t in $[t_0, t_1]$

$X[t_1, t_2]$ ↙ ↘ $X[t_2, t_3]$ ↙ ↘

$X[t_0, t_2]$ $X[t_1, t_3]$ $X[t_2, t_4]$

$VR(X[t_1, t_2], \varepsilon)$ ↙ ↘ $VR(X[t_2, t_3], \varepsilon)$ ↙ ↘

$VR(X[t_0, t_2], \varepsilon)$ $VR(X[t_1, t_3], \varepsilon)$ $VR(X[t_2, t_4], \varepsilon)$

$C^0 \leftarrow C^1 \rightarrow C^2 \leftarrow C^3 \rightarrow C^4$

$$C^0 \leftarrow C^1 \rightarrow C^2 \leftarrow C^3 \rightarrow C^4$$

$$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$$

Persistent Homology:

$$C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow C^3 \rightarrow C^4$$

$$H^0 \rightarrow H^1 \rightarrow H^2 \rightarrow H^3 \rightarrow H^4$$

$$H_k^{i,\,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i) = L(i, i+p)(H_k^i)$$

---

Zigzag Homology:

$$C^0 \leftarrow C^1 \rightarrow C^2 \leftarrow C^3 \rightarrow C^4$$

$$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$$

$$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$$

$$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$$

$$Z_2 \leftarrow Z_2 \rightarrow 0 \leftarrow 0 \rightarrow 0$$

$$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$$

$$Z_2 \leftarrow Z_2 \rightarrow 0 \leftarrow 0 \rightarrow 0$$

$$0 \leftarrow Z_2 \rightarrow Z_2 \leftarrow Z_2 \rightarrow 0$$

$$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$$

$$Z_2 \leftarrow Z_2 \rightarrow 0 \leftarrow 0 \rightarrow 0$$

$$\otimes$$

$$0 \leftarrow Z_2 \rightarrow Z_2 \leftarrow Z_2 \rightarrow 0$$

$$Z_2 \leftarrow Z_2 \otimes Z_2 \rightarrow Z_2 \leftarrow Z_2 \rightarrow 0$$

Gabriel (1972) For Dynkin-Coxeter graphs:

$H^0 \leftarrow H^1 \rightarrow H^2 \leftarrow H^3 \rightarrow H^4$

$\mathbf{F} \leftarrow \mathbf{F} \rightarrow 0 \leftarrow 0 \rightarrow 0$

$\otimes$

$0 \leftarrow \mathbf{F} \rightarrow \mathbf{F} \leftarrow \mathbf{F} \rightarrow 0$

$\mathbf{F} \leftarrow \mathbf{F}\otimes\mathbf{F} \rightarrow \mathbf{F} \leftarrow \mathbf{F} \rightarrow 0$

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$\mathbf{F} \leftarrow \mathbf{F} \rightarrow 0 \leftarrow 0 \rightarrow 0$

$\otimes$

$0 \leftarrow \mathbf{F} \rightarrow \mathbf{F} \leftarrow \mathbf{F} \rightarrow 0$

$\mathbf{F} \leftarrow \mathbf{F}\otimes\mathbf{F} \rightarrow \mathbf{F} \leftarrow \mathbf{F} \rightarrow 0$

I(0, 2)   I(1, 4)

---

**Codensity**

For integer $k > 0$, and PCD $\mathbb{X}$

$$\delta_k(x) = d(x, x')$$

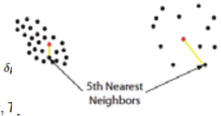
$$x' = \text{ any } k\text{-th nearest neighbor to } x \in \mathbb{X}$$

$$\delta_k(x) \text{ large} \Longrightarrow x \text{ is sparse}$$

$$\delta_k(x) \text{ small} \Longrightarrow x \text{ is dense}$$

$\mathcal{M}[k, T]$ is $T$% densest points as measured by $\delta_k$

What is the persistent homology of these $\mathcal{M}[k, T]$

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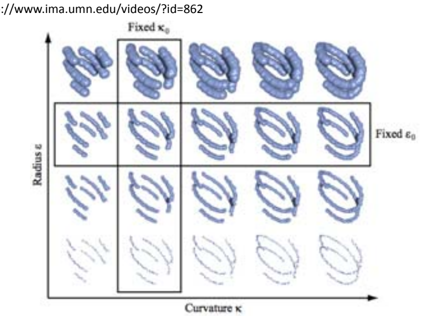
The Theory of Multidimensional Persistence, Gunnar Carlsson, Afra Zomorodian

**"Persistence and Point Clouds" Functoriality, diagrams, difficulties in classifying diagrams, multidimensional persistence, Gröbner bases,** Gunnar Carlsson
http://www.ima.umn.edu/videos/?id=862

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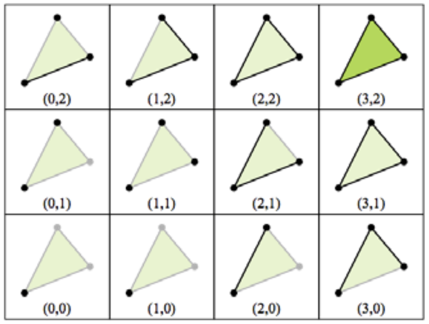


**Figure 2.** A bifiltration of a triangle.

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## Persistence

- Inclusions $VR(X, \epsilon) \hookrightarrow VR(X, \epsilon')$ induce linear transformations

$$H_k(VR(X, \epsilon)) \longrightarrow H_k(VR(X, \epsilon'))$$

- Constructs a *persistence vector space*, i.e. a family $\{V_\epsilon\}_\epsilon$ together with linear transformations

$$L(\epsilon, \epsilon') : V_\epsilon \to V_{\epsilon'}$$

such that

$$L(\epsilon'', \epsilon') \circ L(\epsilon', \epsilon) = L(\epsilon'', \epsilon)$$

when $\epsilon \leq \epsilon' \leq \epsilon''$.

From **Gunnar Carlsson,** Lecture 7: Persistent Homology,
http://www.ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-

## Persistence

- Persistence vector spaces form a *category*, i.e. there are appropriate notions of morphisms of them
- A *morphism* of persistence vector spaces from $\{V_\epsilon\}_\epsilon \to \{W_\epsilon\}_\epsilon$ is a family of linear transformations $f_\epsilon : V_\epsilon \to W_\epsilon$ such that the diagrams

$$\begin{array}{ccc} V_\epsilon & \xrightarrow{f_\epsilon} & W_\epsilon \\ {\scriptstyle L^V(\epsilon,\epsilon')}\downarrow & & \downarrow{\scriptstyle L^W(\epsilon,\epsilon')} \\ V_{\epsilon'} & \xrightarrow{f_{\epsilon'}} & W_{\epsilon'} \end{array}$$

all commute

From **Gunnar Carlsson,** Lecture 7: Persistent Homology, http://www.ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-

- First, restrict $\epsilon$ to a discrete set within the positive real line, i.e. numbers of the form $k\epsilon_0$, where $\epsilon_0$ is a small positive number. Call such an object a $\delta$-persistence vector space.
- This means that a $\delta$-persistence vector space can now we thought of as a *diagram* of the form

$$V_0 \xrightarrow{L(0,1)} V_1 \xrightarrow{L(1,2)} V_2 \xrightarrow{L(2,3)} \cdots$$

- $\delta$-persistence vector spaces also form a category in their own right

From **Gunnar Carlsson,** Lecture 7: Persistent Homology, http://www.ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-

## Persistence

- A $\delta$-persistence vector space is said to be *of finite type* if
  1. All the vector spaces $V_i$ are finite dimensional
  2. There is an integer $N$ so that for all $i \geq N$, we have that $L(i, i+1)$ is an invertible linear transformation
- Condition is analogous to finite dimensionality condition on vector spaces
- We will find a classification theorem for $\delta$-persistence vector spaces of finite type

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## Persistence

1. Given $\{V_i\}_{i\geq 0}$, construct a graded $k[t]$-module $\mu(\{V_i\}_{i\geq 0})$ (whose underlying $k$-vector space is

$$\bigoplus_{i=0}^{\infty} V_i$$

2. The $k$-module structure is the infinite direct sum module structure, i.e. $\kappa \cdot (v_0, v_1, v_2, \ldots) = (\kappa \cdot v_0, \kappa \cdot v_1, \kappa \cdot v_2, \ldots)$
3. $t$ acts by $t \cdot (v_0, v_1, v_2, \ldots) = (0, L(0,1)(v_0), L(1,2)(v_1, \ldots)$
4. Extend action to whole ring using distributivity

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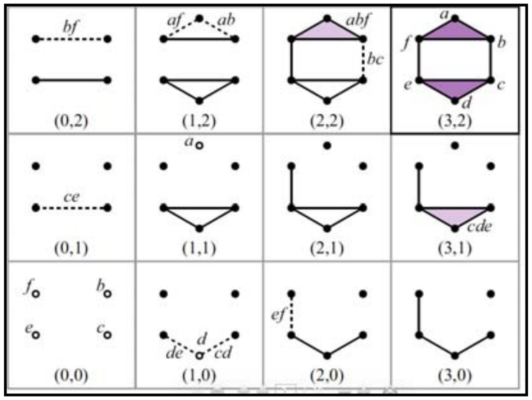
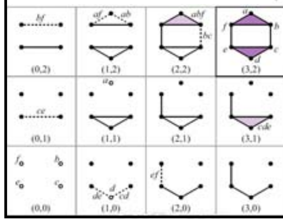



| | | ab $x_1x_2^2$ | bc $x_1^2x_2^2$ | cd $x_1$ | de $x_1$ | ef $x_1^2$ | af $x_1x_2^2$ | bf $x_2^2$ | ce $x_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | $x_1x_2$ | $x_2$ | 0 | 0 | 0 | 0 | $x_2$ | 0 | 0 |
| $d$ | $x_1$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $b$ | 1 | $x_1x_2^2$ | $x_1^2x_2^2$ | 0 | 0 | 0 | 0 | $x_2^2$ | 0 |
| $c$ | 1 | 0 | $x_1^2x_2^2$ | $x_1$ | 0 | 0 | 0 | 0 | $x_2$ |
| $e$ | 1 | 0 | 0 | 0 | $x_1$ | $x_1^2$ | 0 | 0 | $x_2$ |
| $f$ | 1 | 0 | 0 | 0 | 0 | $x_1^2$ | $x_1x_2^2$ | $x_2^2$ | 0 |



Computing Multidimensional Persistence, Gunnar Carlsson, Gurjeet Singh, and Afra Zomorodian